THEORY OF EQUATIONS

POLYNOMIAL FUNCTIONS

Definition:

A function defined by

$$f(x) = a_0x^n + a_1x^{n-1} + ... + a_n, \quad where \quad a_o \neq 0,$$ n is a non negative integer and a_i (i = 0, 1….,n) are fixed complex numbers is called a **polynomial** of **degree** n in x. Then numbers $a_o, a_1, ..., a_n$ are called the **coefficients** of f.

If α is a complex number such that f(α)= 0, then α is called **zero** of the polynomial.

Theorem

Every polynomial of degree n has n and only n zeroes.

Proof:

Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$, *where* $a_o \neq 0$, be a polynomial of degree n $\geq$ 1.

By fundamental theorem of algebra, f(x) has at least one zero, let α_1 be that zero.

Then $(x-\alpha_1)$ is a factor of f(x).

Therefore, we can write:

f(x) = $(x-\alpha_1)Q_1(x)$, where $Q_1(x)$ is a polynomial function of degree n - 1.

If n -1 $\geq$ 1, again by Fundamental Theorem of Algebra, Q_1 (x) has at least one zero, say α_2.

Therefore, $f(x) = (x-\alpha_1)(x-\alpha_2)Q_2(x)$ *where* $Q_2(x)$ is a polynomial function of degree n - 2.

Repeating the above arguments, we get

$f(x) = (x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)Q_n(x)$, where Q_n (x) is a polynomial function of degree n - n = 0, i.e., $Q_n(x)$ is a constant.

Equating the coefficient of x^n on both sides of the above equation, we get

$Q_n(x) = a_o$.

Therefore, $f(x) = a_o(x-\alpha_1)(x-\alpha_2)....(x-\alpha_n)$.

If α is any number other than $\alpha_1, \alpha_2, ...\alpha_n$, then $f(x) \neq 0 \Rightarrow \alpha$ is not a zero of f(x).

Hence f(x) has n and only n zeros, namely $\alpha_1, \alpha_2, ..., \alpha_n$.

Note:

Let $f(x) = a_o x^n + a_1 x^{n-1} + ... + a_n ; a_o \neq 0$ be an n^{th} degree polynomial in x.

Then, $a_o x^n + a_1 x^{n-1} + ... + a_n = 0$ --------- (1)

is called a **polynomial equation** in x of degree n.

A number α is called a **root** of the equation (1) if α is a zero of the polynomial f(x).

Solved Problems

1. Solve $x^4 - 4x^2 + 8x + 35 = 0$, given $2 + i\sqrt{3}$ is a root.

Solution :

Given that $2 + i\sqrt{3}$ is a root of $x^4 - 4x^2 + 8x + 35 = 0$; since complex roots occurs in conjugate pairs $2 - i\sqrt{3}$ is also a root of it.

$\Rightarrow [x - (2 + i\sqrt{3})][x - (2 - i\sqrt{3})] = (x-2)^2 + 3 = x^2 - 4x + 7$ is a factor of the given polynomial.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 + 4x + 5$.

The roots of $x^2 + 4x + 5 = 0$ are given by $\frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$.

Hence the roots of the given polynomial are $2 + i\sqrt{3},\ 2 - i\sqrt{3},\ -2 + i$ *and* $-2 - i$.

2. Solve $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$, given that one of the roots is $1-\sqrt{5}$.

Solution:

Since quadratic surds occur in conjugate pairs as roots of a polynomial equation, $1+\sqrt{5}$ is also a root of the given polynomial.

$\Rightarrow [x-(1-\sqrt{5})][x-(1+\sqrt{5})] = (x-1)^2 - 5 = x^2 - 2x - 4$ is a factor.

Dividing the given polynomial by this factor, we obtain the other factor as $x^2 - 3x + 2$.

Also, $x^2 - 3x + 2 = (x - 2)(x - 1)$

Thus the roots of the given polynomial equation are $1+\sqrt{5}, 1-\sqrt{5}, 1, 2$.

Relation between the Roots and Coefficients of a Polynomial Equation

If α and β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$, then $\alpha + \beta = \frac{-b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

If α and β and γ are the roots of $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$, then $\alpha + \beta + \gamma = \frac{-b}{a}$,

and $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$ *and* $\alpha\beta\gamma = \frac{-d}{a}$.

Illustrative Examples:

1. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression, show that $2p^3 - 9pq + 27r = 0$.

Solution:

Let the roots of the given equation be $a - d$, a, $a + d$.

Then $S_1 = a - d + a + a + d = 3a = -p \Rightarrow a = \frac{-p}{3}$

Since a is a root, it satisfies the given polynomial

$$\Rightarrow \left(-\frac{p}{3}\right)^3 + p.\left(-\frac{p}{3}\right)^2 + q.\left(-\frac{p}{3}\right) + r = 0$$

On simplification, we obtain $2p^3 - 9pq + 27r = 0$.

Symmetric Functions of the Roots

Consider the expressions like $\alpha^2+\beta^2+\gamma^2, (\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2$, $(\beta+\gamma)(\gamma+\alpha)(\alpha-\beta)$. Each of these expressions is a function of α, β, γ with the property that if any two of α, β, γ are interchanged, the function remains unchanged.

Such functions are called **symmetric functions**.

1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of the following in terms of the coefficients.

(i) $\Sigma \frac{1}{\beta\gamma}$ (ii) $\Sigma \frac{1}{\alpha}$ (iii) $\Sigma \alpha^2\beta$

Solution:

Here $\alpha + \beta + \gamma = -p$, $\alpha\beta + \beta\gamma + \alpha\gamma = q$, $\alpha\beta\gamma = -r$

(i) $$\sum \frac{1}{\beta\gamma} = \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}$$

(ii) $$\sum \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma} = \frac{q}{-r} = -\frac{q}{r}$$

(iii) $$\Sigma \alpha^2\beta = \alpha^2\beta + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + \alpha^2\gamma + \beta^2\gamma$$

$$= (\alpha\beta + \beta\gamma + \alpha\gamma)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma = (q \,.\, -p) - 3\,(-r) = 3r - pq \,.$$

THANK YOU