MATHEMATICAL ANALYSIS

MODULE- 2

OPEN SETS, CLOSED SETS AND COUNTABLE SETS

NEIGHBOURHOOD OF A POINT

A set $N \subseteq R$ is called the neighbourhood of a point a, if there exists an open interval I containing a and contained in N, i.e.,

$$a \in I \subseteq N$$

An open interval is a neighbourhood of each of its points. We shall take the open interval $(a-\delta,\ a+\delta)$ where $\delta > 0$ as a neighbourhood of the point a.

DELETED NEIGHBOURHOODS

The set $\{x : 0 < |x-a| < \delta\}$ i.e., an open interval $(a-\delta,\ a+\delta)$ from which the number a itself has been excluded or deleted is called a deleted neighbourhood of a.

ILLUSTRATIONS

1. The set **R** of real numbers is the neighbourhood of each of its points.
2. The set **Q** of rationals is not the nbd of any of its points.
3. The open interval (a, b) is nbd of each of its points.
4. The closed interval $[a, b]$ is the nbd of each point of (a, b) but is not a nbd of the end points a and b.
5. The null set ϕ is a nbd of each of its points in the sense that there is no point in ϕ of which it is not a nbd.

EXAMPLE 1. A non- empty finite set is not a nbd of any point.

A set can be a nbd of a point if it contains an open interval containing the point. Since an interval necessarily contains an infinite number of points, therefore, in order that a set be a nbd of a point it must necessarily contain an infinity of points. Thus a finite set cannot be a nbd of any point.

EXAMPLE 2. Superset of a nbd of a point x is also a nbd of x. i.e., if N is a nbd of a point x and $M \supseteq N$ then M is also a nbd of x.

EXAMPLE 3. Union (finite or arbitrary) of nbds of a point x is again a nbd of x.

EXAMPLE 4. If M and N are nbds of a point x, then show that $M \cap N$ is also a nbd of x.

Soln. Since M , N are nbds of x, ∃ open intervals enclosing the point x such that

$x \in (x-\delta_1, x+\delta_1) \subseteq M$ and $x \in (x-\delta_2, x+\delta_2) \subseteq N$

Let $\delta = \min(\delta_1, \delta_2)$. Then

$(x-\delta, x+\delta) \subseteq (x-\delta_1, x+\delta_1) \subseteq M$ and $(x-\delta, x+\delta) \subseteq (x-\delta_2, x+\delta_2) \subseteq N$

$\Rightarrow (x-\delta, x+\delta) \subseteq M \cap N$

$\Rightarrow x \in (x-\delta, x+\delta) \subseteq M \cap N$

$\Rightarrow M \cap N$ is a nbd of x.

INTERIOR POINTS OF A SET

A point x is an interior point of a set S if S is a nbd of x. In other words x is an interior point of S if ∃ an open interval (a,b) containing x and contained in S, i.e., $x \in (a,b) \subseteq S$. Thus a set is a nbd of each of its interior points.

Interior of a Set. The set of all interior points of a set is called the interior of the set. The interior of a set is generally denoted by S' or int S.

Ex. The interior of the set **N** or **I** or **Q** is the null set, but interior of **R** is **R**.

The interior of a set S is a subset of S. i.e., int S $\subseteq$ S.

[$x \in \text{int} S \Rightarrow x \in (x-\delta, x+\delta) \subseteq S \Rightarrow x \in S$. i.e., int S $\subseteq$ S.]

OPEN SET

A set S is said to be open if it is a nbd of each of its points, i.e., for each $x \in S$, there exist an open interval I_x such that

$$x \in I_x \subseteq S.$$

Thus every point of an open set is an interior point, so that for an open set S, int S= S.

Evidently, S is open $\Leftrightarrow$ S=int S.

The set is not open if it is not a nbd of at least one of its points, or that there is at least one point of the set which is not an interior point.

ILLUSTRATIONS

1. The set **R** of real numbers is an open set.

2. The set **Q** of rationals is not an open set.

3. The closed interval $[a, b]$, is not open for it is not a neighbourhood of the end points a and b.

4. The null set ϕ is open, for there is no point in ϕ of which it is not a neighbourhood.

5. A non-empty finite set is not open.

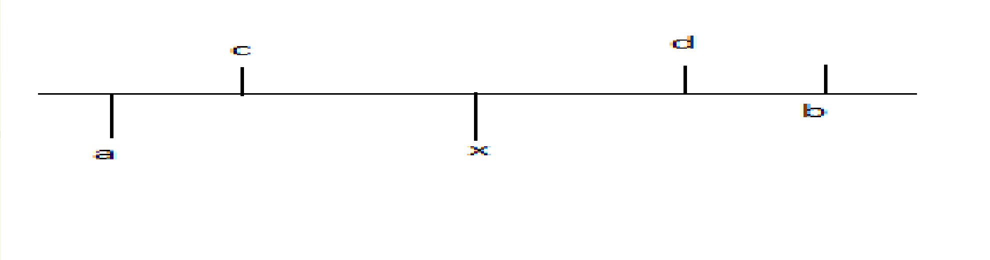
6. The set $\{\frac{1}{n} : n \in N\}$ is not open.

EXAMPLE 5. Show that every open interval is an open set.

Or

Every open inerval is a nbd of each of its points.

Let x be any point of the given open interval (a, b) so that we have $a < x < b$.



Let c, d be two numbers such that

$$a < c < x \text{ and } x < d < b$$

so that we have

$$a < c < x < d < b \Rightarrow x \in (c, d) \subset (a, b).$$

Thus the given interval (a, b) contains an open interval containing the point x, and is therefore a nbd of x.

Hence, the open interval is a nbd of each of its points and is therefore an open set.

Example 6. Show that every open set is a union of open intervals .

Let S be an open set and x_λ a point of S .

Since S is open, therefore ∃ an open interval I_{x_λ} for each of its points x_λ such that

$$x_\lambda \in I_{x_\lambda} \subseteq S \quad \forall x_\lambda \in S$$

Again the set S can be thought of as the union of singleton sets like $\{x_\lambda\}$, i.e.,

$S = \bigcup_{\lambda\in\Lambda} \{x_\lambda\}$, where Λ is the index set

$$\therefore S = \bigcup_{\lambda\in\Lambda} \{x_\lambda\} \subseteq \bigcup_{\lambda\in\Lambda} I_{x_\lambda} \subseteq S$$

$$\Rightarrow S = \bigcup_{\lambda\in\Lambda} I_{x_\lambda}$$

THEOREM 1: The interior of a set is an open set.

Let S be a given set, and $\operatorname{int} S$ its interior.

If $\operatorname{int} S = \phi$ then $\operatorname{int} S$ is open.

When $\operatorname{int} S \neq \phi$, let $x \in \operatorname{int} S$.

As x is an interior point of S, there exist an open interval I_x such that $x \in I_x \subseteq S$

But I_x being an open interval, is a nbd of each of its points .

⇨ Every point of I_x is an interior point of I_x, and
$I_x \subseteq S \Rightarrow$ every point of I_x is an interior point of S
$\therefore I_x \subseteq \operatorname{int} S$

⇨ $x \in I_x \subseteq \operatorname{int} S \Rightarrow$ any point of x of $\operatorname{int} S$ is interior point of $\operatorname{int} S$

⇨ $\operatorname{int} S$ is an open set.

Corollary: The interior of a set S is an open subset of S.

THEOREM 2: The interior of a set S is the largest open subset of S.

Or

The interior of a set S contains every open subset of S.

We know that the interior $\operatorname{int} S$ of a set S is an open subset of S. Let us show that any open subset S_1 of S is contained in $\operatorname{int} S$.

Let $x \in S_1$.

Since an open set is a nbd of each of its points, therefore S_1 is a nbd of x.

But S is a superset of S_1.

$\therefore$ S is also a nbd of x

⇨ x is an interior point of S

⇨ $x \in \operatorname{int} S$

Thus

⇨ $x \in S_1 \Rightarrow x \in \operatorname{int} S$

$\therefore$ $S_1 \subseteq \operatorname{int} S$

Hence, every open subset of S is contained in its interior $\operatorname{int} S$.

$\Rightarrow \operatorname{int} S$, the interior of S is the largest open subset of S.

Corollary: Interior of the set S is the union of all open subsets of S.

THEOREM 3: The union of an arbitrary family of open sets is open.

Let F be the union of an arbitrary family $\mathcal{F} = \{S_\lambda : \lambda \in \Lambda\}$ of open sets, Λ being an index set. To prove that F is open, we shall show that for any point $x \in$ F, it contains an open interval containing x.

Let $x \in$ F. Since F is the union of the members of $\mathcal{F}$, ∃ at least one member, say S_λ of $\mathcal{F}$ which contains x. Again, S_λ being an open set, there exist an open interval I_x such that

$x \in I_x \subseteq S_\lambda \subseteq$ F.

Thus the set F contains an open interval containing any point x of F

$\Rightarrow$ F is an open set.

THEOREM 4: The intersection of any finite number of open sets is open.

Let us consider two open sets S, T.

If S ∩ T = ∅, it is an open set.

If S ∩ T ≠ ∅, let $x \in$ S ∩ T.

Now

$x \in$ S ∩ T $\Rightarrow x \in$ S $\wedge$ $x \in$ T

⇒S, T are nbds of x [since S, T are open]

⇒ S ∩ T is a nbd of x.

But since x is any point of S ∩ T, therefore S ∩ T is a nbd of each of its points.

Hence S ∩ T is open.

This proof may of course be extended to a finite number of sets.

Note: The above theorem does not hold for the intersection of arbitrary family of open sets.

Consider for example the open sets $S_n = (\frac{-1}{n}, \frac{1}{n}), \; n \in N$

Their intersection is the set {0} consisting of a single point 0, and this set is not open.

LIMIT POINTS OF A SET

Definition 1: A real number ξ is a limit point of a set S (⊂R) if every nbd of ξ contains an infinite number of members of S.

Thus ξ is a limit point of set S if for any nbd N of ξ N ∩S is an infinite set.

A limit point is also called a cluster point, a condensation point or an accumulation point.

A limit point of a set may or may not be a member of the set. A set may have no limit point, a unique limit point, a finite or an infinite number of limit points.

Definition 2: A real number ξ is a limit point of a set S (⊆ R) if every nbd of ξ contains atleast one member of S other than ξ.

A point ξ is not a limit point of a set S if ∃ even one nbd of ξ not containing any point of S other than ξ.

Derived Sets: The set of all the limit points of a set S is called the derived set of *S* and is denoted by *S′*.

ILLUSTRATIONS

1. The set I has no limit point, for a nbd $(m-\frac{1}{2}, m+\frac{1}{2})$ of m ∈ I, contains no point of I other than m. Thus the derived set of I is the null set ϕ.

2. Every point of R is a limit point, for, every nbd of any of its points contains an infinity of members of R .Therefore R’=R

3. Every point of the set Q of rationals is a limit point of Q, for, between any two rationals there exists an infinity of rationals. Further every irrational number is also a limit point of Q for between any two irrationals there are infinitely many rationals. Thus every real number is a limit point of Q, so that Q’=R.

4. The set $\{\frac{1}{n} : n \in N\}$ has only one limit point, zero, which is not a member of the set.

5. Every point of the closed interval [a, b] is its limit point, and a point not belonging to the interval is not a limit point. Thus the derived set [a, b]’= [a, b].

6. Every point of the (a, b) is its limit point. The end points a, b which are not members of (a, b) are also its limit points. Thus (a, b)’=[a, b].

A finite set has no limit point. An infinite set may or may not have limit points.

Bolzano- Weierstrass Theorem(for sets)

Every infinite bounded set has a limit point.

Let S be any infinite bounded set and m, M its infimum and supremum respectively. Let P be a set of real numbers defined as follows:

x ∈P iff it exceeds at the most a finite number of members of S.

The set P is non empty, for m∈P. Also M is an upper bound of P, for no number greater than or equal to M can belong to P. Thus the set P is non-empty and is bounded above .Therefore by the order-completeness property, P has the supremum, say ξ. We shall now show that ξ is a limit point of S.

Consider any nbd (ξ -ε, ξ + ε) of ξ, where ε>0.

Since ξ is the supremum of P ,∃ at least one member say η of P such that η > ξ - ε. Now η belongs to P, therefore it exceeds at the most a finite number of members of S, and consequently ξ - ε (<η) can exceed at the most a finite number of members of S.

Again as ξ is the supremum of P, ξ + ε cannot belong to P, and consequently ξ + ε must exceed an infinite number of members of S.

Now ξ - ε exceeds at the most a finite number of members of S and ξ + ε exceeds infinitely members of S.

⇨ (ξ - ε, ξ + ε) contains an infinite number of members of S

Consequently ξ is a limit point of S.

Note: Boundedness is not necessary in order for an infinite set S to have a limit point. The set S= {½, 2, ⅓, 3……} is unbounded and infinite and has the limit point 0. The unbounded interval (a, ∞) has infinitely many limit points.

Examples:

1. If S and T are subsets of real numbers, then show that

1. $S \subseteq T \Rightarrow S' \subseteq T'$
2. $(S \cup T)' = S' \cup T'$

Soln:

1. If $S' = \phi$ then evidently $S' \subseteq T'$.

When $S' \neq \phi$, let $\xi \in S'$ and N be any nbd of ξ.

⇨ N contains an infinite number of members of S.

But $S \subseteq T$, ∴ N contains infinitely many members of T

⇨ ξ is limit point of T, i.e., $\xi \in T'$

Thus $\xi \in S' \Rightarrow \xi \in T'$. Hence $S' \subseteq T'$.

2. Now $S \subseteq S \cup T \Rightarrow S' \subseteq (S \cup T)'$ and

$T \subseteq S \cup T \Rightarrow T' \subseteq (S \cup T)'$

Consequently,

$S' \cup T' \subseteq (S \cup T)'$ (1)

Now we proceed to show that $(S \cup T)' \subseteq S' \cup T'$.

If $(S \cup T)' = \phi$, then evidently $(S \cup T)' \subseteq S' \cup T'$

When $(S \cup T)' \neq \phi$, let $\xi \in (S \cup T)'$

Now ξ is a limit points of S U T, therefore, every nbd of ξ contains infinitely many points of S or T or Both.

- ⇨ ξ is a limit point of S or a limit point of T
- ⇨ $\xi \in S' \vee \xi \in T'$
- ⇨ $\xi \in S' \cup T'$

Thus, $\xi \in (S \cup T)' \Rightarrow \xi \in S' \cup T'$

Consequently,

$(S \cup T)' \subseteq S' \cup T'$ (2)

From (1) and (2) it follows that

$(S \cup T)' = S' \cup T'$

Thus the derived set of the union= the union of the derived sets.

THANK YOU