Binomial Distribution

Binomial Experiment

1. The **binomial experiment** consists of a fixed number of trials: *n*
2. Each trial has two possible outcomes: *success* and *failure*.
3. The probability of success is *p*. The probability of failure is 1 – *p*.
4. The trials are independent.

Binomial random variable is the number of successes in *n* trials.

Each trial is a **Bernoulli process** if properties 2 – 4 are satisfied.

Binomial experiment?

- Flip a coin 10 times.
- Draw 5 cards out of a shuffled deck.
- A political survey asks 1500 voters whom they intend to vote.

Binomial Probability Distribution

Example: What is the probability of getting 2 heads when a fair coin is flipped 4 times?

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Solution: *HHTT*, *HTHT*, *HTTH*, *THHT*, *THTH*, *TTHH*

So there are 6 ways to get 2 heads in 4 flips. ($C_2^4 = 6$)

Each sequence has probability $(0.5)^2(0.5)^2$.

P(2 heads in 4 flips) = $6(0.5)^2(0.5)^2 = .375$

Binomial Probability Distribution

x = number of successes in binomial experiment with n trials.
x takes values 0, 1, 2, … , n, therefore, it is discrete.
$n - x$ = number of failures.

Probability that there are x successes and $n - x$ failures:

$$p^x(1-p)^{n-x}$$

Number of ways to get x successes and $n - x$ failures:

$$C_x^n = \frac{n!}{x!(n-x)!}$$

where $n! = n(n-1)(n-2) \cdot \cdot \cdot (2)(1)$, e.g. $0!=1$, $3!=3(2)(1)=6$.

Binomial Probability Distribution

The probability of *x* successes in a binomial experiment with *n* trials and probability of success *p* is

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \text{ for } x = 0, 1, \ldots, n$$

Example: A quiz consists of 10 multiple-choice questions. Each question has 5 possible answers, only one of which is correct. Pat plans to guess the answer to each question. Find the probability that Pat gets

a. one answer correct.

b. all 10 answers correct.

Solution: $n = 10$, $p = .2$

a.
$$P(1) = \frac{10!}{1!(10-1)!}(.2)^1(1-.2)^{10-1} = 10(.2)(.8)^9 = .2684$$

b.
$$P(10) = \frac{10!}{10!(10-10)!}(.2)^{10}(1-.2)^{10-10} = 1(.2)^{10}(1) = .0000001$$

Cumulative Probability

$$P(X \leq x) = P(0) + P(1) + \ldots + P(x)$$

Example: Find the probability that Pat fails the quiz. A mark is considered a failure if it is less than 50%.

Solution: A mark of less than 5 is a failure.

$$\begin{aligned} P(X \leq 4) &= P(0) + P(1) + P(2) + P(3) + P(4) \\ &= .1074 + .2684 + .3020 + .2013 + .0881 \\ &= .9672 \end{aligned}$$

Using the cumulative probability

$$P(X \geq x) = 1 - P(X \leq x - 1)$$

Example: Find the probability that Pat passes the quiz.

Solution: A mark of 5 or greater is a pass.

$$\begin{aligned} P(X \geq 5) &= 1 - P(X \leq 4) \\ &= 1 - .9672 \\ &= .0328 \end{aligned}$$

Using the cumulative probability

$$P(X = x) = P(X \le x) - P(X \le x - 1)$$

Example: Find the probability that Pat gets one answer correct.

Solution:

$$\begin{aligned} P(1) &= P(X \le 1) - P(X \le 0) \\ &= .3758 - .1074 \\ &= .2684 \end{aligned}$$

Using the cumulative probability

$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a - 1)$$

Example: Find the probability that Pat gets 5 or 6 answers correct.

Solution:

$$P(5 \leq X \leq 6) = P(X \leq 6) - P(X \leq 4)$$
$$= .9991 - .9672$$
$$= .0319$$

(or $P(5 \leq X \leq 6) = P(5) + P(6) = .0264 + .0055 = .0319$)

Mean and Variance of a Binomial Distribution

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

$$\sigma = \sqrt{np(1 - p)}$$

Example: If a class is full of students like Pat, what is the mean mark? What is the standard deviation?

Solution:

$$\mu = np = 10(0.2) = 2$$

$$\sigma = \sqrt{np(1 - p)} = \sqrt{10(.2)(1 - .2)} = 1.26$$

Poisson Distribution

Poisson Experiment

1. The number of successes that occur in any interval is independent of the number of successes that occur in any other intervals.
2. The probability of a success in an interval is the same for all equal-size intervals.
3. The probability of a success in an interval is proportional to the size of the interval.
4. The probability of more than one success in an interval approaches 0 as the interval becomes smaller.

Poisson Random Variable

The **Poisson random variable** is the number of successes that occur in a period of time or an interval of space in a Poisson experiment.

As a general rule, a Poisson random variable is the number of *relatively rare* event that occurs *randomly* and *independently*.

Example: (not Poisson random variables)

- The number of hits on an active website
- The number of people arriving at a restaurant

Poisson Random Variable

Example: (Poisson Random Variables)

- The number of cars arriving at a service station in 1 hour
- The number of flaws in a bolt of cloth
- The number of accidents in 1 day on a particular stretch of highway

Poisson Probability Distribution

The probability that a Poisson random variable assumes a value of *x* is

$$P(x) = \frac{e^{-\mu}\mu^{x}}{x!} \text{ for } x = 0, 1, 2, \ldots$$

where μ is the mean number of successes in the interval or region and e is the base of the natural logarithm (approx. 2.71828).

The variance of the Poisson r.v.

$$\sigma^2 = \mu$$

Example: The number of arrivals at a car wash is Poisson distributed with a mean of 8 per hour. What is the probability that

a. no car will arrive in the next hour?

b. at most 8 cars will arrive in the next hour?

Solution: Use $\mu = 8$.

a. $P(0) = e^{-8}8^0/0! = .0003$

b. $P(X \leq 8) = P(0) + P(1) + \ldots + P(8)$

Using cumulative probability in MiniTab gives

$P(X \leq 8) = .5925$

Example: The number of arrivals at a car wash is Poisson distributed with a mean of 8 per hour. What is the probability that

a. no car will arrive in the next 2 hours?

b. at most 10 cars will arrive in the next 2 hours?

c. at least 15 cars will arrive in the next 2 hours?

Solution: The mean number of car arrivals in 2 hour period is 16. Thus, use $\mu = 16$.

a. $P(0) = e^{-16}16^0/0! = 0.0000001$

b. $P(X \leq 10) = 0.0774$

c. $P(X \geq 15) = 1 - P(X \leq 14) = 1 - .3675 = .6325$

Example: The number of arrivals at a car wash is Poisson distributed with a mean of 8 per hour. What is the probability that

a. exactly 15 cars will arrive in the next 2 hours?

b. between 10 to 20 cars will arrive in the next 2 hours?

Solution: $\mu = 16$.

a. $P(15) = P(X \leq 15) - P(X \leq 14)$

$= .4667 - .3675 = .0992$

b. $P(10 \leq X \leq 20) = P(X \leq 20) - P(X \leq 9)$

$= .8682 - .0433 = .8249$

Recall $P(a \leq X \leq b) = P(X \leq b) - P(X \leq a - 1)$