CYLINDRICAL AND SPHERICAL COORDINATES

Cylindrical Coordinates

In the cylindrical coordinate system, a point in space (Figure 12.7.1) is represented by the ordered triple (r, θ, z), where

- (r, θ) are the polar coordinates of the point's projection in the xy-plane
- z is the usual z-**coordinate** in the Cartesian coordinate system

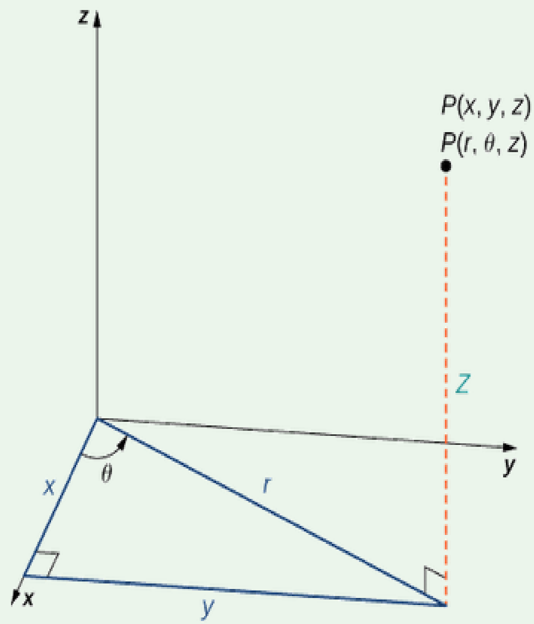

Figure 12.7.1: The right triangle lies in the xy-plane. The length of the hypotenuse is r and θ is the measure of the angle formed by the positive x-axis and the hypotenuse. The z-coordinate describes the location of the point above or below the xy-plane.

Conversion between Cylindrical and Cartesian Coordinates

The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related as follows:

These equations are used to convert from cylindrical coordinates to rectangular coordinates.

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

These equations are used to convert from rectangular coordinates to cylindrical coordinates

1. $r^2 = x^2 + y^2$
2. $\tan\theta = \dfrac{y}{x}$
3. $z = z$

SPHERICAL COORDINATES

In the *spherical coordinate system*, a point P in space (Figure 11.7.9) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates;
- φ (the Greek letter phi) is the angle formed by the positive z-axis and line segment $\bar{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.

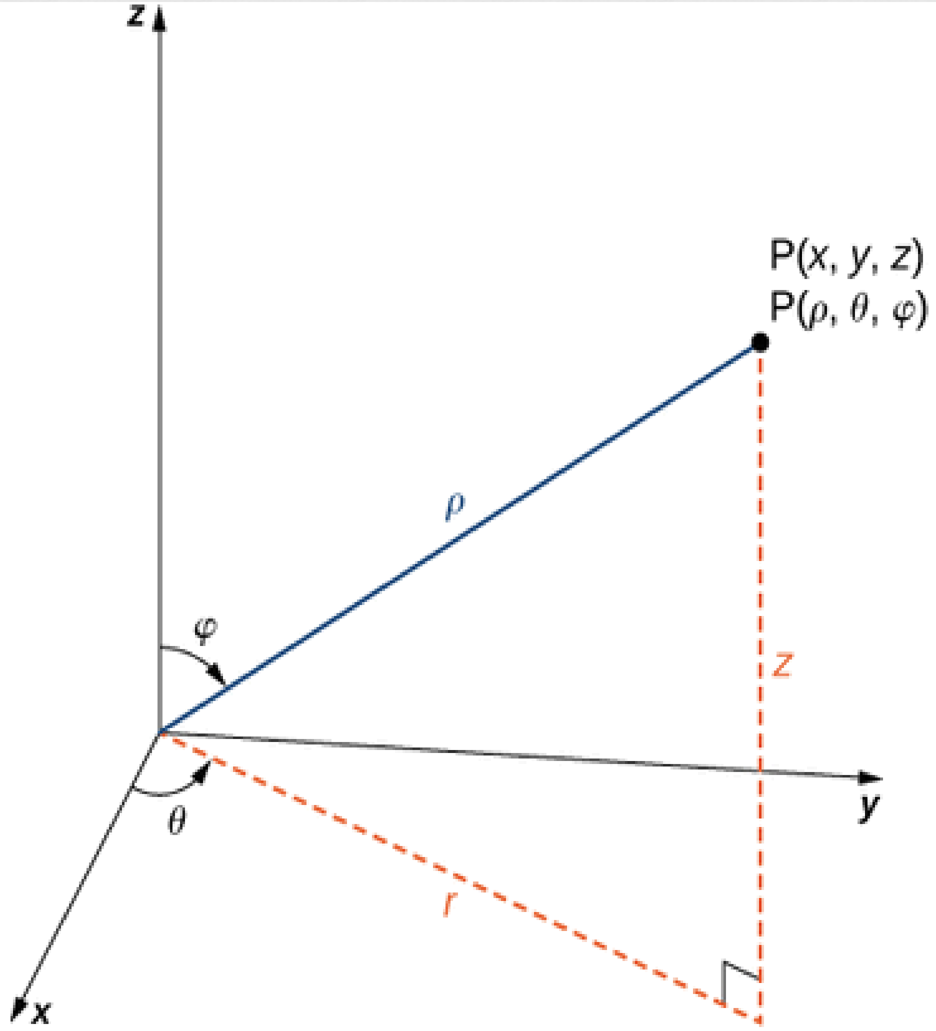

Figure 11.7.9: The relationship among spherical, rectangular, and cylindrical coordinates.

HOWTO: Converting among Spherical, Cylindrical, and Rectangular Coordinates

Rectangular coordinates (x, y, z), cylindrical coordinates (r, θ, z), and spherical coordinates (ρ, θ, φ) of a point are related as follows:

Convert from spherical coordinates to rectangular coordinates

These equations are used to convert from spherical coordinates to rectangular coordinates.

- $x = \rho \sin \varphi \cos \theta$
- $y = \rho \sin \varphi \sin \theta$
- $z = \rho \cos \varphi$

Convert from rectangular coordinates to spherical coordinates

These equations are used to convert from rectangular coordinates to spherical coordinates.

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan \theta = \dfrac{y}{x}$
- $\varphi = \arccos(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}})$.

Convert from spherical coordinates to cylindrical coordinates

These equations are used to convert from spherical coordinates to cylindrical coordinates.

- $r = \rho \sin \varphi$
- $\theta = \theta$
- $z = \rho \cos \varphi$

Convert from cylindrical coordinates to spherical coordinates

These equations are used to convert from cylindrical coordinates to spherical coordinates.

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$

PROBLEMS

Convert the rectangular coordinates $(1, -3, 5)$ to cylindrical coordinates.

Solution

Use the second set of equations from Note to translate from rectangular to cylindrical coordinates:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \pm\sqrt{1^2 + (-3)^2} \\ &= \pm\sqrt{10}. \end{aligned}$$

We choose the positive square root, so $r = \sqrt{10}$.Now, we apply the formula to find θ. In this case, y is negative and x is positive, which means we must select the value of θ between $\frac{3\pi}{2}$ and 2π:

$$\begin{aligned} \tan\theta &= \frac{y}{x} & &= \frac{-3}{1} \\ \theta &= \arctan(-3) & &\approx 5.03\,\text{rad}. \end{aligned}$$

In this case, the z-coordinates are the same in both rectangular and cylindrical coordinates:

$$z = 5.$$

The point with rectangular coordinates $(1, -3, 5)$ has cylindrical coordinates approximately equal to $(\sqrt{10}, 5.03, 5)$.

PROBLEMS

Plot the point with spherical coordinates $(8, \frac{\pi}{3}, \frac{\pi}{6})$ and express its location in both rectangular and cylindrical coordinates.

Solution

Use the equations in Note to translate between spherical and cylindrical coordinates (Figure 11.7.12):

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ &= 8\sin(\frac{\pi}{6})\cos(\frac{\pi}{3}) \\ &= 8(\frac{1}{2})\frac{1}{2} \\ &= 2 \\ y &= \rho \sin\varphi \sin\theta \\ &= 8\sin(\frac{\pi}{6})\sin(\frac{\pi}{3}) \\ &= 8(\frac{1}{2})\frac{\sqrt{3}}{2} \\ &= 2\sqrt{3} \\ z &= \rho\cos\varphi \\ &= 8\cos(\frac{\pi}{6}) \\ &= 8(\frac{\sqrt{3}}{2}) \\ &= 4\sqrt{3} \end{aligned}$$

PROBLEMS

$$r = \rho \sin \varphi$$

$$= 8 \sin \frac{\pi}{6} \qquad = 4$$

$$\theta = \theta$$

$$z = \rho \cos \varphi$$

$$= 8 \cos \frac{\pi}{6}$$

$$= 4\sqrt{3}.$$

Thus, cylindrical coordinates for the point are $(4, \frac{\pi}{3}, 4\sqrt{3})$.

PROBLEMS

Convert the rectangular coordinates $(-1, 1, \sqrt{6})$ to both spherical and cylindrical coordinates.

Solution

Start by converting from rectangular to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2 = (-1)^2 + 1^2 + (\sqrt{6})^2 = 8$$

$$\tan\theta = \frac{1}{-1}$$

$$\rho = 2\sqrt{2} \text{ and } \theta = \arctan(-1) = \frac{3\pi}{4}.$$

Because $(x, y) = (-1, 1)$, then the correct choice for θ is $\frac{3\pi}{4}$.

There are actually two ways to identify φ. We can use the equation $\varphi = \arccos(\frac{z}{\sqrt{x^2 + y^2 + z^2}})$. A more simple approach, however, is to use equation $z = \rho\cos\varphi$. We know that $z = \sqrt{6}$ and $\rho = 2\sqrt{2}$, so

$$\sqrt{6} = 2\sqrt{2}\cos\varphi, \text{ so } \cos\varphi = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

and therefore $\varphi = \frac{\pi}{6}$. The spherical coordinates of the point are $(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6})$.

To find the cylindrical coordinates for the point, we need only find r:

$$r = \rho\sin\varphi = 2\sqrt{2}\sin(\frac{\pi}{6}) = \sqrt{2}.$$

The cylindrical coordinates for the point are $(\sqrt{2}, \frac{3\pi}{4}, \sqrt{6})$.

THANK YOU