SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

Second-Order Linear Equations

- A **second-order linear differential equation** has the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

- where *P*, *Q*, *R*, and *G* are continuous functions.

- In this section we study the case where $G(x) = 0$, for all x, in Equation 1.

- Such equations are called **homogeneous** linear equations.

Second-Order Linear Equations

- Thus the form of a second-order linear homogeneous differential equation is

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

- If $G(x) \neq 0$ for some x, Equation 1 is **nonhomogeneous**.

Second-Order Linear Equations

- Two basic facts enable us to solve homogeneous linear equations.
- The first of these says that if we know two solutions y_1 and y_2 of such an equation, then the **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution

3 Theorem If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation 2 and c_1 and c_2 are any constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of Equation 2.

Second-Order Linear Equations

- The other fact we need is given by the following theorem, which is proved in more advanced courses.

  It says that the general solution is a linear combination of two **linearly independent** solutions y_1 and y_2.

  This means that neither y_1 nor y_2 is a constant multiple of the other.

  For instance, the functions $f(x) = x^2$ and $g(x) = 5x^2$ are linearly dependent, but $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent.

Second-Order Linear Equations

4 Theorem If y_1 and y_2 are linearly independent solutions of Equation 2 on an interval, and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants.

- Theorem 4 is very useful because it says that if we know *two* particular linearly independent solutions, then we know *every* solution.

  In general, it's not easy to discover particular solutions to a second-order linear equation.

Second-Order Linear Equations

- But it is always possible to do so if the coefficient functions
  P, *Q*, and *R* are constant functions, that is, if the differential equation has the form

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$$ay'' + by' + cy = 0$$

- where *a*, *b*, and *c* are constants and $a \neq 0$.

Second-Order Linear Equations

- We are looking for a function y such that a constant times its second derivative y'' plus another constant times y' plus a third constant times y is equal to 0.

  We know that the exponential function $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself: $y' = re^{rx}$. Furthermore, $y'' = r^2e^{rx}$. If we substitute these expressions into Equation 5, we see that
  $y = e^{rx}$ is a solution if

- $$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$
- or
- $$(ar^2 + br + c)e^{rx} = 0$$

Second-Order Linear Equations

- But e^{rx} is never 0. Thus $y = e^{rx}$ is a solution of Equation 5 if r is a root of the equation

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$$ar^2 + br + c = 0$$

- Equation 6 is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$.

- Notice that it is an algebraic equation that is obtained from the differential equation by replacing y'' by r^2, y' by r, and y by 1.

Second-Order Linear Equations

8 If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

10 If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

11 If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Second-Order Linear Equations

- Solve the equation $y'' + y' - 6y = 0$.
- Solution:
  The auxiliary equation is

$$r^2 + r - 6 = (r - 2)(r + 3) = 0$$

- whose roots are $r = 2, -3$.
- Therefore, by , the general solution of the given differential equation is
- $$y = c_1 e^{2x} + c_2 e^{-3x}$$

Second-Order Linear Equations

- Solve the equation $4y'' + 12y' + 9y = 0$.

- **Solution:**
  The auxiliary equation is $4r^2 + 12r + 9 = 0$ can be factored as

  $$(2r + 3)^2 = 0$$

- so the only root is $r = -3/2$ By eqn (10) , the general solution is

- $$y = c_1 e^{-3x/2} + c_2 x e^{-3x/2}$$

Second-Order Linear Equations

- Solve the equation $y'' - 6y' + 13y = 0$.

- **Solution:**
  The auxiliary equation is $r^2 - 6r + 13 = 0$. By the quadratic formula, the roots are

$$r = \frac{6 \pm \sqrt{36 - 52}}{2}$$
$$= \frac{6 \pm \sqrt{-16}}{2}$$
$$= 3 \pm 2i$$

the general solution of the differential equation is

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

THANK YOU