

PROPOSITIONAL LOGIC

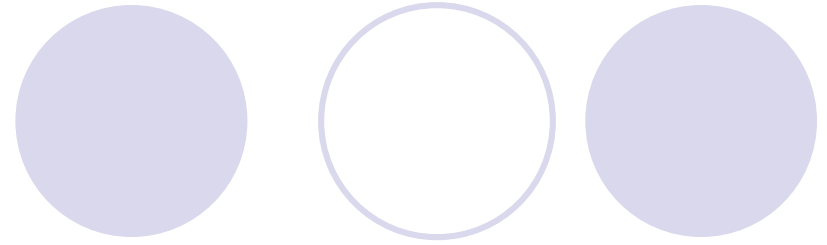
Discrete Mathematics and Its Applications



Introduction

- A **proposition** is a **declarative** sentence (a sentence that declares a fact) that is either **true or false**, but not both.
- Are the following sentences propositions?
 - Toronto is the capital of Canada. (Yes)
 - Read this carefully. (No)
 - $1+2=3$ (Yes)
 - $x+1=2$ (No)
 - What time is it? (No)

Propositional Logic



- **Propositional Logic** – the area of logic that deals with propositions
- **Propositional Variables** – variables that represent propositions: p, q, r, s
 - E.g. Proposition p – “Today is Friday.”
- **Truth values** – T, F

Propositional Logic

DEFINITION 1

Let p be a proposition. The negation of p , denoted by $\neg p$, is the statement “It is not the case that p .”

The proposition $\neg p$ is read “not p .” The truth value of the negation of p , $\neg p$ is the opposite of the truth value of p .

● Examples

- Find the negation of the proposition “Today is Friday.” and express this in simple English.

Solution: The negation is “It is not the case that *today is Friday*.”
In simple English, “Today is not Friday.” or “It is not Friday today.”

- Find the negation of the proposition “At least 10 inches of rain fell today in Miami.” and express this in simple English.

Solution: The negation is “It is not the case that *at least 10 inches of rain fell today in Miami*.”
In simple English, “Less than 10 inches of rain fell today in Miami.”

Propositional Logic

- Note: Always assume fixed times, fixed places, and particular people unless otherwise noted.

- Truth table:

The Truth Table for the Negation of a Proposition.	
p	$\neg p$
T	F
F	T

- **Logical operators** are used to form new propositions from two or more existing propositions. The logical operators are also called **connectives**.

Propositional Logic

DEFINITION 2

Let p and q be propositions. The *conjunction* of p and q , denoted by $p \wedge q$, is the proposition “ p and q ”. The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

● Examples

- Find the conjunction of the propositions p and q where p is the proposition “Today is Friday.” and q is the proposition “It is raining today.”, and the truth value of the conjunction.

Solution: The conjunction is the proposition “Today is Friday and it is raining today.” The proposition is true on rainy Fridays.

Propositional Logic

DEFINITION 3

Let p and q be propositions. The *disjunction* of p and q , denoted by $p \vee q$, is the proposition “ p or q ”. The conjunction $p \wedge q$ is false when both p and q are false and is true otherwise.

- Note:

inclusive or: The disjunction is true when at least one of the two propositions is true.

- E.g. “Students who have taken calculus or computer science can take this class.” – those who take one or both classes.

exclusive or: The disjunction is true only when one of the proposition is true.

- E.g. “Students who have taken calculus or computer science, but not both, can take this class.” – only those who take one of them.

- Definition 3 uses *inclusive or*.

Propositional Logic

DEFINITION 4

Let p and q be propositions. The *exclusive or* of p and q , denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The Truth Table for the Exclusive Or (XOR) of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Propositional Logic

Conditional Statements

DEFINITION 5

Let p and q be propositions. The *conditional statement* $p \rightarrow q$, is the proposition “if p , then q .” The conditional statement is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

- A conditional statement is also called an implication.
- Example: “If I am elected, then I will lower taxes.” $p \rightarrow q$

implication:

elected, lower taxes.	T	T		T
not elected, lower taxes.	F	T		T
not elected, not lower taxes.	F	F		T
elected, not lower taxes.	T	F		F

Propositional Logic

- Example:

- Let p be the statement “Maria learns discrete mathematics.” and q the statement “Maria will find a good job.” Express the statement $p \rightarrow q$ as a statement in English.

Solution: Any of the following -

“If Maria learns discrete mathematics, then she will find a good job.”

“Maria will find a good job when she learns discrete mathematics.”

“For Maria to get a good job, it is sufficient for her to learn discrete mathematics.”

“Maria will find a good job unless she does not learn discrete mathematics.”

Propositional Logic

- Other conditional statements:

- **Converse** of $p \rightarrow q : q \rightarrow p$

- **Contrapositive** of $p \rightarrow q : \neg q \rightarrow \neg p$

- **Inverse** of $p \rightarrow q : \neg p \rightarrow \neg q$

Propositional Logic

DEFINITION 6

Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition “ p if and only if q .” The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

- $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$
 - “*if and only if*” can be expressed by “*iff*”
 - Example:
 - Let p be the statement “You can take the flight” and let q be the statement “You buy a ticket.” Then $p \leftrightarrow q$ is the statement “You can take the flight if and only if you buy a ticket.”
- Implication:**
- If you buy a ticket you can take the flight.
If you don't buy a ticket you cannot take the flight.

Propositional Logic

The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Propositional Logic

Truth Tables of Compound Propositions

- We can use connectives to build up complicated compound propositions involving any number of propositional variables, then use truth tables to determine the truth value of these compound propositions.
- Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.					
p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

1.1 Propositional Logic

Precedence of Logical Operators

- We can use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
\neg	1
\wedge	2
\vee	3
\rightarrow	4
\leftrightarrow	5

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$$p \wedge q \vee r = (p \wedge q) \vee r$$

$$p \vee q \wedge r = p \vee (q \wedge r)$$

Propositional Logic

Translating English Sentences

- English (and every other human language) is often ambiguous. Translating sentences into compound statements removes the ambiguity.
- Example: How can this English sentence be translated into a logical expression?

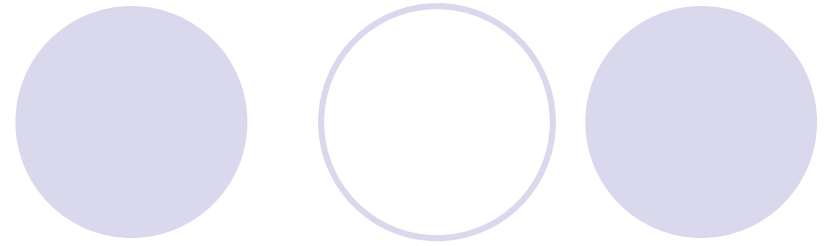
“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”

Solution: Let q , r , and s represent “You can ride the roller coaster,”

“You are under 4 feet tall,” and “You are older than 16 years old.” The sentence can be translated into:

$$(r \wedge \neg s) \rightarrow \neg q.$$

Propositional Logic



- Example: How can this English sentence be translated into a logical expression?

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

Solution: Let a , c , and f represent “You can access the Internet from campus,” “You are a computer science major,” and “You are a freshman.” The sentence can be translated into:

$$a \rightarrow (c \vee \neg f).$$

Propositional Logic

Logic and Bit Operations

- Computers represent information using bits.
- A **bit** is a symbol with two possible values, 0 and 1.
- By convention, 1 represents T (true) and 0 represents F (false).
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation – replace true by 1 and false by 0 in logical operations.

Table for the Bit Operators <i>OR</i> , <i>AND</i> , and <i>XOR</i> .				
x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Propositional Logic

DEFINITION 7

A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

- Example: Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit string 01 1011 0110 and 11 0001 1101.

Solution:

01 1011 0110	
11 0001 1101	

11 1011 1111	bitwise <i>OR</i>
01 0001 0100	bitwise <i>AND</i>
10 1010 1011	bitwise <i>XOR</i>

Propositional Equivalences

Introduction

DEFINITION 1

A compound proposition that is always true, no matter what the truth values of the propositions that occurs in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology or a contradiction is called a *contingency*.

Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Propositional Equivalences

Logical Equivalences

DEFINITION 2

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.
- Example: Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Propositional Equivalences

Constructing New Logical Equivalences

- Example: Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution:

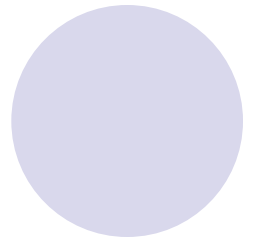
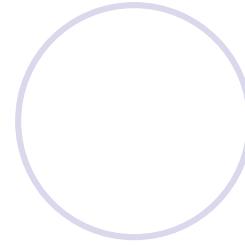
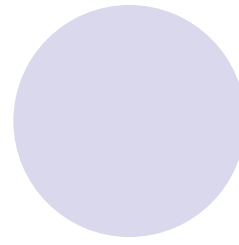
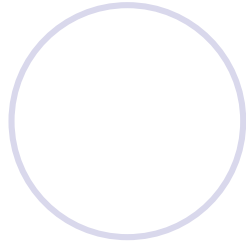
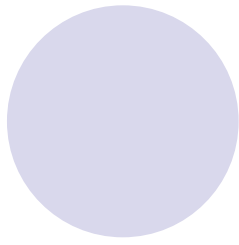
$$\begin{aligned}\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by example on slide 21} \\ &\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan law} \\ &\equiv p \wedge \neg q && \text{by the double negation law}\end{aligned}$$

- Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

$$\begin{aligned}(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by example on slide 21} \\ &\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and} \\ &&& \text{communicative law for disjunction} \\ &\equiv T \vee T \\ &\equiv T\end{aligned}$$

- Note: The above examples can also be done using truth tables.



THANK YOU