PARTIAL ORDERINGS

Partial Orderings: Definitions

- **Definitions**:
  - A relation *R* on a set *S* is called a <u>partial order</u> if it is
    - Reflexive
    - Antisymmetric
    - Transitive
  - A set S together with a partial ordering R is called a <u>partially ordered set</u> (poset, for short) and is denote (*S*,*R*)

Partial Orderings: Notation

- We use the notation:
  - $a \prec b$, when $(a,b) \in R$
  - $a \prec b$, when $(a,b) \in R$ and $a \neq b$
- The notation $\prec$ is not to be mistaken for "less than"
- The notation $\prec$ is used to denote <u>any</u> partial ordering

Comparability: Definition

- **Definition**:
  - The elements a and b of a poset $(S, \prec)$ are called <u>comparable</u> if either $a \prec b$ or $b \prec a$.
  - When for $a,b \in S$, we have neither $a \prec b$ nor $b \prec a$, we say that a,b are <u>incomparable</u>

Total orders: Definition

- **Definition**:
  - If $(S,\prec)$ is a poset and every two elements of S are comparable, S is called a totally ordered set.
  - The relation $\prec$ is said to be a total order
- Example
  - The relation "less than or equal to" over the set of integers $(\mathbb{Z}, \leq)$ since for every $a,b \in \mathbb{Z}$, it must be the case that $a \leq b$ or $b \leq a$
  - What happens if we replace $\leq$ with $<$?
- The relation $<$ is not reflexive, and $(\mathbb{Z},<)$ is not a poset

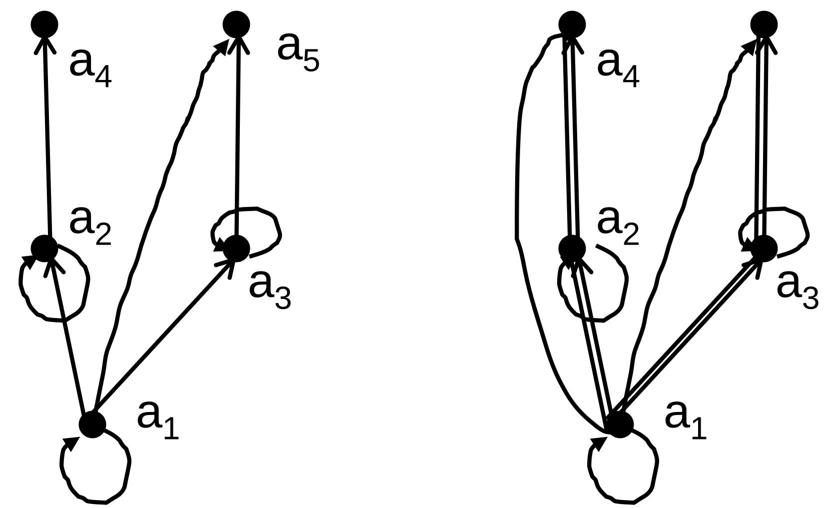
Well Orderings: Definition

- **Definition**: $(S,\prec)$ is a well-ordered set if
  - It is a poset
  - Such that $\prec$ is a total ordering and
  - Such that every non-empty subset of S has a least element
- Example
  - The natural numbers along with $\leq$, $(\mathbb{N},\leq)$, is a well-ordered set since any nonempty subset of $\mathbb{N}$ has a least element and $\leq$ is a total ordering on $\mathbb{N}$
  - However, $(\mathbb{Z},\leq)$ is not a well-ordered set
    - Why?
    - Is it totally ordered?
- $\mathbb{Z}^{-} \subset \mathbb{Z}$ but does not have a least element
- Yes

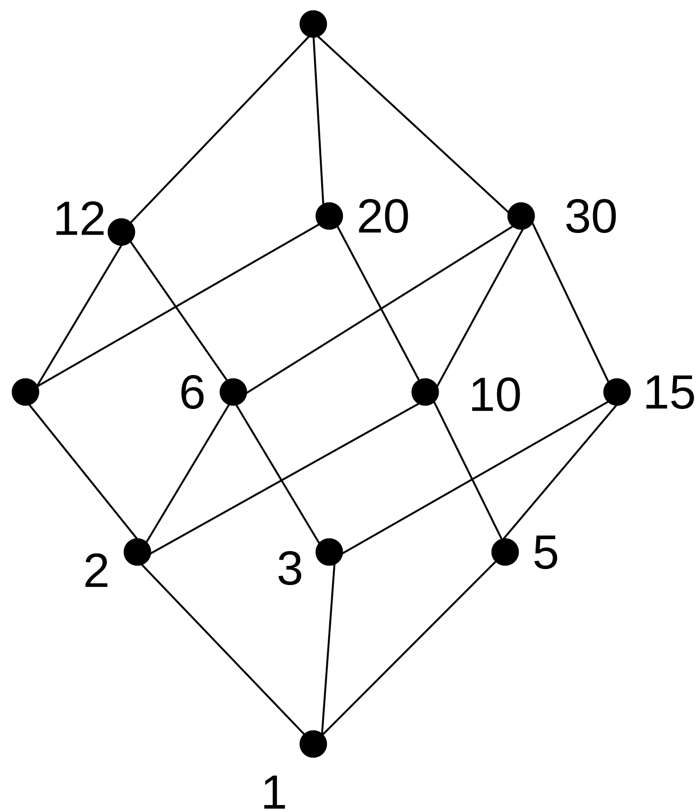
Hasse Diagrams

- Like relations and functions, partial orders have a convenient graphical representation: Hasse Diagrams
  - Consider the digraph representation of a partial order
  - Because we are dealing with a partial order, we know that the relation must be reflexive and transitive
  - Thus, we can simplify the graph as follows
    - Remove all self loops
    - Remove all transitive edges
    - Remove directions on edges assuming that they are oriented upwards
  - The resulting diagram is far simpler

Hasse Diagram: Example



Hasse Diagram: Example (2)



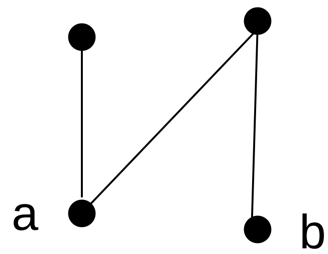
Extremal Elements: Maximal

- **Definition**: An element a in a poset (S, $\prec$) is called maximal if it is not less than any other element in S. That is: $\neg(\exists b \in S\ (a \prec b))$
- If there is one unique maximal element a, we call it the maximum element (or the greatest element)

Extremal Elements: Minimal

- **Definition**: An element a in a poset (S, $\prec$) is called minimal if it is not greater than any other element in S. That is: $\neg(\exists b \in S\ (b \prec a))$
- If there is one unique minimal element a, we call it the minimum element (or the least element)

Extremal Elements: Example 1



- What are the minimal, maximal, minimum, maximum elements?
- Minimal: {a,b}
- Maximal: {c,d}
- There are no unique minimal or maximal elements, thus no minimum or maximum

THANK YOU