Fourier series of odd and even functions

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FOURIER SERIES

JOSEPH FOURIER

(Founder of Fourier series)

Fourier Series

FOURIER SERIES can be generally written as,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

Where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \quad \text{.........} \ \mathbf{(1.1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx \quad \text{.........} \ \mathbf{(1.2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx \quad \text{.........} \ \mathbf{(1.3)}$$

BASIS FORMULAE OF FOURIER SERIES

- The Fourier series of a periodic function $f(x)$ with period 2π is defined as the trigonometric series with the coefficient a0, an and bn, known as FOURIER COEFFICIENTS, determined by formulae (1.1), (1.2) and (1.3).

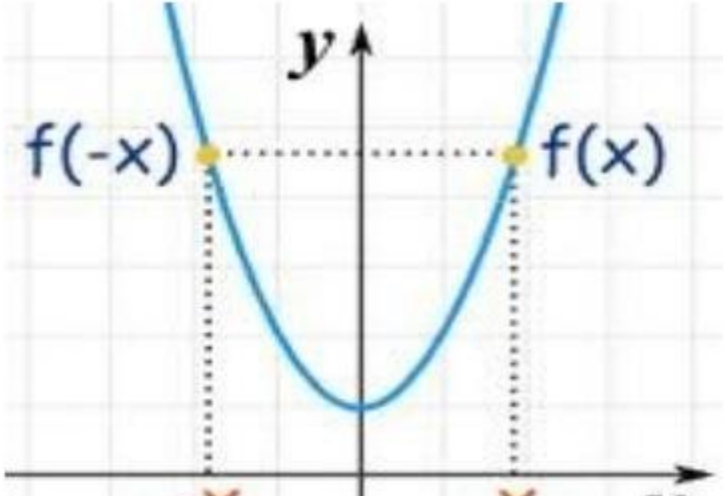
- REMEMBER THIS SECTION IS APPLICABLE ONLY WHEN $f(x)$ IS DEFINED IN THE INTERVAL $[-\pi, \pi] or \ [-l, l]$

Even Functions

- Definition: A function f(x) is said to be even if $f(-x)=f(x)$.

  e.g. x^2, cosx are even function

  Graphically, an even function is symmetrical about y-axis.

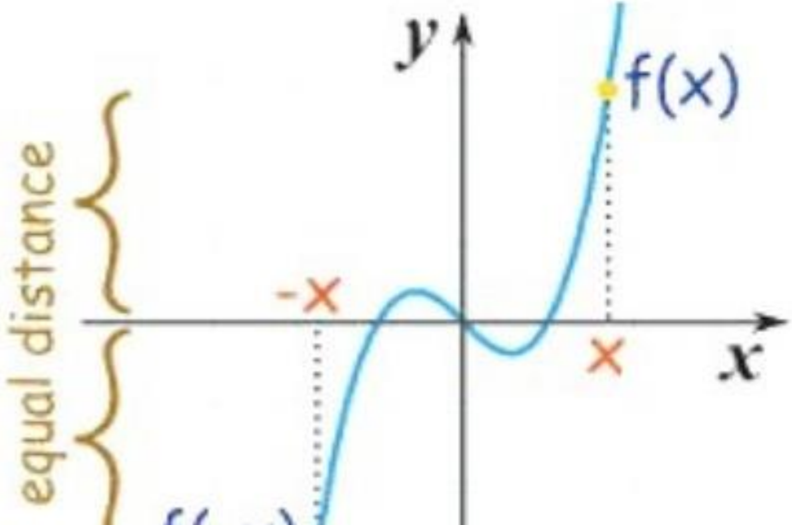


Odd Functions

- Definition: A function f(x) is said to be odd if f(-x)=-f(x).

  e.g. sin x, x^3 are odd functions.

  Graphically, an even function is symmetrical about the origion.



Important result

- 1) For even function, g(x), $\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$
- (2) For odd function, h(x), $\int_{-L}^{L} h(x)dx = 0$

Even Functions

- When function is even:

When f(x) is an even function then f(x)sinx is an odd function.

Thus $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$

$$ao = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

$$bn = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0$$

Therefore $f(x) = \frac{a0}{2} + \sum_{n=1}^{\alpha} an\, cosnx$

Odd function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)sinnx\,dx$$

1. The product of two even functions or two odd funnctions is an even function. Whereas the product of an even function and an odd function is an odd function.
2. The given function not necessary even or odd or it may be neither. Every function can be uniquely decomposed into the sum of an even function, and an odd function.

that is $f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=f_1(x)+f_2(x)$

Where $f_1(x)$ is even function and $f_2(x)$is odd function.

(b) Find the Fourier Series for $f(x) = x \sin x,\ -\pi \le x \le \pi$. Deduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots = \frac{\pi - 2}{4}.$$

Solution : Here $f(x)$ is an even function.

$\therefore\ b_n = 0$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$\therefore\ a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx$$

$$= \frac{2}{\pi}\left[(x)(-\cos x) - (1)(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\pi = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,\{\sin(n+1)x - \sin(n-1)x\}\,dx$$

$$= \frac{1}{\pi}\left[(x)\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\}\right]$$

$$= (-1)^{n+1}\left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\} = \frac{2(-1)^{n+1}}{n^2-1}, \quad n \ne 1$$

For $n = 1$,

Put $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(n-1)(n+1)} \cos\frac{n\pi}{2}$$

$$\therefore \frac{\pi}{2} - 1 = 2\left\{+\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\right\}$$

$$\therefore \frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

(b) Express $f(x) = x \cos x$ as a Fourier Series in $(-\pi, \pi)$.

Solution : Here $f(x)$ is an odd function.

$$\therefore \quad a_0 = a_n = 0$$

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$\therefore \; b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \cos x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \{\sin (n+1) x + \sin (n-1) x\} \, dx$$

$$= \frac{1}{\pi}\left[(x)\left\{-\frac{\cos (n+1) x}{n+1} - \frac{\cos (n-1) x}{n-1}\right\} - (1)\left\{-\frac{\sin (n+1) x}{(n+1)^2} - \frac{\sin (n-1) x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right\}\right] = -(-1)^{n+1}\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= -\frac{2n(-1)^{n+1}}{n^2-1}, \quad n \neq 1$$

For $n = 1$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$\therefore \quad x\cos x = b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -\frac{1}{2}\sin x - 2\sum_{n=2}^{\infty} \frac{n(-1)^{n+1}}{n^2-1}\sin nx$$

- $\boldsymbol{F(x) = x + x^2}$ is neither even nor odd

- REMEMBER THIS SECTION IS APPLICABLE ONLY WHEN $f(x)$ IS DEFINED IN THE INTERVAL $[-\pi, \pi]$ or $[-l, l]$


Thank You!