FOURIER SERIES AND TRANSFORM

SANIL JOSE

DEPARTMENT OF MATHEMATICS

SACRED HEART COLLEGE

Theorem 11.15 (Fejér). *Assume that $f \in L([0, 2\pi])$ and suppose that f is periodic with period 2π. Define a function s by the following equation:*

$$s(x) = \lim_{t \to 0+} \frac{f(x + t) + f(x - t)}{2}, \tag{27}$$

whenever the limit exists. Then, for each x for which $s(x)$ is defined, the Fourier series generated by f is Cesàro summable and has (C, 1) sum $s(x)$. That is, we have

$$\lim_{n \to \infty} \sigma_n(x) = s(x),$$

where $\{\sigma_n\}$ is the sequence of arithmetic means defined by (23). If, in addition, f is continuous on $[0, 2\pi]$, then the sequence $\{\sigma_n\}$ converges uniformly to f on $[0, 2\pi]$.

11.15 THE WEIERSTRASS APPROXIMATION THEOREM

Fejér's theorem can also be used to prove a famous theorem of Weierstrass which states that every continuous function on a compact interval can be uniformly approximated by a polynomial. More precisely, we have:

Theorem 11.17. *Let f be real-valued and continuous on a compact interval $[a, b]$. Then for every $\varepsilon > 0$ there is a polynomial p (which may depend on ε) such that*

$$|f(x) - p(x)| < \varepsilon \qquad \text{for every } x \text{ in } [a, b]. \tag{30}$$

Proof. If $t \varepsilon [0, \pi)$, let $g(t) = f[a + t(b - a)/\pi]$; if $t \in [\pi, 2\pi]$, let $g(t) = f[a + (2\pi - t)(b - a)/\pi]$ and define g outside $[0, 2\pi]$ so that g has period 2π. For the ε given in the theorem, we can apply Fejér's theorem to find a function σ defined by an equation of the form

$$\sigma(t) = A_0 + \sum_{k=1}^{N} (A_k \cos kt + B_k \sin kt)$$

such that $|g(t) - \sigma(t)| < \varepsilon/2$ for every t in $[0, 2\pi]$. (Note that N, and hence σ, depends on ε.) Since σ is a finite sum of trigonometric functions, it generates a power series expansion about the origin which converges uniformly on every finite interval. The partial sums of this power series expansion constitute a sequence of polynomials, say $\{p_n\}$, such that $p_n \to \sigma$ uniformly on $[0, 2\pi]$. Hence, for the same ε, there exists an m such that

$$|p_m(t) - \sigma(t)| < \frac{\varepsilon}{2}, \qquad \text{for every } t \text{ in } [0, 2\pi].$$

We have $|P_m(t) - g(t)| = |P_m(t) - \sigma(t) + \sigma(t) - g(t)|$

$$\leq |P_m(t) - \sigma(t)| + |\sigma(t) - g(t)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall t \in [0,2\pi] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now define the polynomial p by the formula $p(x) = p_m(\frac{\pi(x-a)}{b-a})$

Put $t = \frac{\pi(x-a)}{b-a}$, we get $g(t) = g\left(\frac{\pi(x-a)}{b-a}\right) = f\left[a + \left(\frac{\pi(x-a)}{b-a}\right)\frac{(b-a)}{\pi}\right] = f(a + x - a)$

$= f(x)$ if $\frac{\pi(x-a)}{b-a} \in [0,\pi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Ie if $\pi(x-a) < \pi(b-a)$

$ie\ x - a < b - a$

$ie\ x < b$

$\therefore$ from (1) ie $|P_m(t) - g(t)| = \varepsilon \; \forall t \in [0,2\pi]$, we get BY (2)

$|p(x) - f(x)| < \varepsilon$

$ie \; |f(x) - p(x)| < \varepsilon \qquad\qquad \forall x \in [a,b]$

HENCE THE THEOREM

11.16 OTHER FORMS OF FOURIER SERIES

Using the formulas

$$2\cos nx = e^{inx} + e^{-inx} \qquad \text{and} \qquad 2i \sin nx = e^{inx} - e^{-inx},$$

the Fourier series generated by f can be expressed in terms of complex exponentials as follows:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\alpha_n e^{inx} + \beta_n e^{-inx}),$$

- For
- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{inx}+e^{-inx}}{2} + b_n \frac{e^{inx}-e^{-inx}}{2i}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{inx}\left(\frac{a_n}{2} - i\frac{b_n}{2}\right) + e^{-inx}\left(\frac{a_n}{2} + i\frac{b_n}{2}\right), \alpha_n = \frac{a_n}{2} - i\frac{b_n}{2}, \beta_n = \frac{a_n}{2} + i\frac{b_n}{2}$

- If we put $\alpha_0 = \frac{a_0}{2}$ and $\alpha_{-n} = \beta_n$, we can write the exponential formula as $f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$, where
- $\alpha_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}[\frac{1}{\pi}\int_0^{2\pi} f(t)cosntdt - i\frac{1}{\pi}\int_0^{2\pi} f(t)sinntdt] = \frac{1}{2\pi}\int_0^{2\pi} f(t)(cosnt - isinnt)dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt \ (n = 0, \pm 1, \pm 2, \ldots\ldots)$

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}.$$

The formulas (7) for the coefficients now become

$$\alpha_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\, dt \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

If f has period 2π, the interval of integration can be replaced by any other interval of length 2π.

More generally, if $f \in L([0, p])$ and if f has period p, we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n x}{p} + b_n \sin \frac{2\pi n x}{p} \right)$$

to mean that the coefficients are given by the formulas

$$a_n = \frac{2}{p} \int_0^p f(t) \cos \frac{2\pi n t}{p} \, dt,$$

$$b_n = \frac{2}{p} \int_0^p f(t) \sin \frac{2\pi n t}{p} \, dt \qquad (n = 0, 1, 2, \ldots).$$

In exponential form we can write

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi i n x/p},$$

where

$$\alpha_n = \frac{1}{p} \int_0^p f(t) e^{-2\pi i n t/p} \, dt, \qquad \text{if } n = 0, \pm 1, \pm 2, \ldots .$$

Theorem 11.18 (Fourier integral theorem). *Assume that $f \in L(-\infty, +\infty)$. Suppose there is a point x in $\mathbf{R}$ and an interval $[x - \delta, x + \delta]$ about x such that either*

a) *f is of bounded variation on $[x - \delta, x + \delta]$,*

or else

b) *both limits $f(x+)$ and $f(x-)$ exist and both Lebesgue integrals*

$$\int_0^\delta \frac{f(x + t) - f(x+)}{t}\, dt \qquad \text{and} \qquad \int_0^\delta \frac{f(x - t) - f(x-)}{t}\, dt$$

exist.

Then we have the formula

$$\frac{f(x+) + f(x-)}{2} = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(u) \cos v(u - x)\, du \right] dv, \tag{32}$$

the integral $\int_0^\infty$ being an improper Riemann integral.

11.8 THE RIEMANN–LEBESGUE LEMMA

Theorem 11.6. *Assume that $f \in L(I)$. Then, for each real β, we have*

$$\lim_{\alpha \to +\infty} \int_I f(t) \sin (\alpha t + \beta)\, dt = 0. \tag{10}$$

Theorem 11.8 (Jordan). *If g is of bounded variation on $[0, \delta]$, then*

$$\lim_{\alpha \to +\infty} \frac{2}{\pi} \int_0^\delta g(t) \frac{\sin \alpha t}{t}\, dt = g(0+). \tag{13}$$

Theorem 11.9 (Dini). *Assume that $g(0+)$ exists and suppose that for some $\delta > 0$ the Lebesgue integral*

$$\int_0^\delta \frac{g(t) - g(0+)}{t}\, dt$$

exists. Then we have

$$\lim_{\alpha \to +\infty} \frac{2}{\pi} \int_0^\delta g(t) \frac{\sin \alpha t}{t}\, dt = g(0+).$$

Proof. The first step in the proof is to establish the following formula:

$$\lim_{\alpha \to +\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \frac{\sin \alpha t}{t} \, dt = \frac{f(x+) + f(x-)}{2}. \tag{33}$$

For this purpose we write

$$\int_{-\infty}^{\infty} f(x+t) \frac{\sin \alpha t}{\pi t} \, dt = \int_{-\infty}^{-\delta} + \int_{-\delta}^{0} + \int_{0}^{\delta} + \int_{\delta}^{\infty}.$$

When $\alpha \to +\infty$, the first and fourth integrals on the right tend to 0, because of the Riemann–Lebesgue lemma. In the third integral, we can apply either Theorem 11.8 or Theorem 11.9 (depending on whether (a) or (b) is satisfied) to get

$$\lim_{\alpha \to +\infty} \int_{0}^{\delta} f(x+t) \frac{\sin \alpha t}{\pi t} \, dt = \frac{f(x+)}{2}.$$

Similarly, we have

$$\int_{-\delta}^{0} f(x+t) \frac{\sin \alpha t}{\pi t} \, dt = \int_{0}^{\delta} f(x-t) \frac{\sin \alpha t}{\pi t} \, dt \to \frac{f(x-)}{2} \qquad \text{as } \alpha \to +\infty.$$

Thus we have established (33). If we make a translation, we get

$$\int_{-\infty}^{\infty} f(x + t)\, \frac{\sin \alpha t}{t}\, dt = \int_{-\infty}^{\infty} f(u)\, \frac{\sin \alpha(u - x)}{u - x}\, du,$$

and if we use the elementary formula

$$\frac{\sin \alpha(u - x)}{u - x} = \int_{0}^{\alpha} \cos v(u - x)\, dv,$$

the limit relation in (33) becomes

$$\lim_{\alpha \to +\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{0}^{\alpha} \cos v(u - x)\, dv \right] du = \frac{f(x+) + f(x-)}{2}. \tag{34}$$

But the formula we seek to prove is (34) with only the order of integration reversed. By Theorem 10.40 we have

$$\int_{0}^{\alpha} \left[\int_{-\infty}^{\infty} f(u) \cos v(u - x)\, du \right] dv = \int_{-\infty}^{\infty} \left[\int_{0}^{\alpha} f(u) \cos v(u - x)\, dv \right] du$$

for every $\alpha > 0$, since the cosine function is everywhere continuous and bounded.

Since the limit in (34) exists, this proves that

$$\lim_{\alpha \to +\infty} \frac{1}{\pi} \int_0^{\alpha} \left[\int_{-\infty}^{\infty} f(u) \cos v(u - x) \, du \right] dv = \frac{f(x+) + f(x-)}{2}.$$

By Theorem 10.40, the integral $\int_{-\infty}^{\infty} f(u) \cos v(u - x) \, du$ is a continuous function of v on $[0, \alpha]$, so the integral $\int_0^{\infty}$ in (32) exists as an improper Riemann integral. It need not exist as a Lebesgue integral.

11.18 THE EXPONENTIAL FORM OF THE FOURIER INTEGRAL THEOREM

Theorem 11.19. *If f satisfies the hypotheses of the Fourier integral theorem, then we have*

$$\frac{f(x+)+f(x-)}{2} = \frac{1}{2\pi} \lim_{\alpha \to +\infty} \int_{-\alpha}^{\alpha} \left[\int_{-\infty}^{\infty} f(u) e^{iv(u-x)}\, du \right] dv. \tag{35}$$

Proof. Let $F(v) = \int_{-\infty}^{\infty} f(u) \cos v(u-x)\, du$. Then F is continuous on $(-\infty, +\infty)$, $F(v) = F(-v)$ and hence $\int_{-\alpha}^{0} F(v)\, dv = \int_0^{\alpha} F(-v)\, dv = \int_0^{\alpha} F(v)\, dv$. Therefore (32) becomes

$$\frac{f(x+)+f(x-)}{2} = \lim_{\alpha \to +\infty} \frac{1}{\pi} \int_0^{\alpha} F(v)\, dv = \lim_{\alpha \to +\infty} \frac{1}{2\pi} \int_{-\alpha}^{\alpha} F(v)\, dv. \tag{36}$$

Now define G on $(-\infty, +\infty)$ by the equation

$$G(v) = \int_{-\infty}^{\infty} f(u) \sin v(u - x)\, du.$$

Then G is everywhere continuous and $G(v) = -G(-v)$. Hence $\int_{-\alpha}^{\alpha} G(v)\, dv = 0$ for every α, so $\lim_{\alpha\to+\infty} \int_{-\alpha}^{\alpha} G(v)\, dv = 0$. Combining this with (36) we find

$$\frac{f(x+) + f(x-)}{2} = \lim_{\alpha\to+\infty} \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \{F(v) + iG(v)\}\, dv.$$

This is formula (35).

11.19 INTEGRAL TRANSFORMS

Many functions in analysis can be expressed as Lebesgue integrals or improper Riemann integrals of the form

$$g(y) = \int_{-\infty}^{\infty} K(x, y) f(x)\, dx. \tag{37}$$

A function g defined by an equation of this sort (in which y may be either real or complex) is called an *integral transform* of f. The function K which appears in the integrand is referred to as the *kernel* of the transform.

Exponential Fourier transform: $\int_{-\infty}^{\infty} e^{-ixy} f(x)\,dx.$

Fourier cosine transform: $\int_{0}^{\infty} \cos xy\, f(x)\,dx.$

Fourier sine transform: $\int_{0}^{\infty} \sin xy\, f(x)\,dx.$

Laplace transform: $\int_{0}^{\infty} e^{-xy} f(x)\,dx.$

Mellin transform: $\int_{0}^{\infty} x^{y-1} f(x)\,dx.$

Hence as $c \to \infty$, (2) reduces to $\boxed{f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda.}$ ...(3)

which is known as **Fourier Integral** of $f(x)$.

Equation (3) is true at a point of continuity. At a point of discontinuity the value of the integral on the right is

$$\frac{1}{2}[f(x+0)+f(x-0)].$$

2.3. FOURIER SINE AND COSINE INTEGRALS

We know that $\cos \lambda(t - x) = \cos \lambda t \cos \lambda x + \sin \lambda t \sin \lambda x$

$\therefore$ Fourier integral of $f(x)$ can be written as

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,\{\cos \lambda t \cos \lambda x + \sin \lambda t \sin \lambda x\}\,dt\,d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty} \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t\,dt\,d\lambda + \frac{1}{\pi}\int_0^{\infty} \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t\,dt\,d\lambda \qquad ...(4)$$

When f(x) is an odd function, $f(t) \cos \lambda t$ is odd while $f(t) \sin \lambda t$ is even. Thus the first integral in (4) vanishes and, we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t\,dt\,d\lambda} \qquad ...(5)$$

This is called **Fourier sine integral.**

When f(x) is an even function, $f(t) \cos \lambda t$ is even while $f(t) \sin \lambda t$ is odd. Thus the second integral in (4) vanishes and, we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty} \cos \lambda x \int_0^{\infty} f(t) \cos \lambda t\,dt\,d\lambda} \qquad ...(6)$$

This is called **Fourier cosine integral.**

Example 1. *Express the function* $f(x) = \begin{cases} 1 & \text{for } |x| \le 1, \\ 0 & \text{for } |x| > 1, \end{cases}$

as a Fourier integral. Hence evaluate $\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$. (M.D.U. May 2008)

Sol. The Fourier integral for $f(x)$ is

$$\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t - x)\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-1}^1 \cos \lambda (t - x)\, dt\, d\lambda \qquad \left[\because f(t) = \begin{cases} 1, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}\right]$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin \lambda(t - x)}{\lambda}\right]_{-1}^1 d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin \lambda(1 - x) - \sin \lambda(-1 - x)}{\lambda} d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin \lambda(1 + x) + \sin \lambda(1 - x)}{\lambda} d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$$

$$\therefore \quad \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

At $|x| = 1$, *i.e.*, $x = \pm 1$, $f(x)$ is discontinuous and the integral has the value

$$\frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \frac{\pi}{4}.$$

Note. Putting $x = 0$, we get $\int_0^\infty \frac{\sin \lambda}{\lambda} d\lambda = \frac{\pi}{2}$ or $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.


Thank You
For Your
Time