DIFFERENTIAL GEOMETRY

Didimos K. V.
Assistant Professor
Department of Mathematics
Sacred Heart College(Autonomous), Thevara, Kochi.

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CONTENTS

Text Book: John A. Thorpe, Elementary Topics in Differential Geometry

Module 1

- Graphs and level sets, vector fields, the tangent space, surfaces, vector fields on surfaces, orientation.
- Chapters 1 to 5 of the text

Module:2

- The Gauss map, geodesics, Parallel transport.
- Chapters 6, 7 & 8 of the text

Module:3

- The Weingarten map, curvature of plane curves, Arc length and line
- Chapters 9, 10 & 11 of the text

Module:4

- Curvature of surfaces, Parametrized surfaces, local equivalence of surfaces and Parametrized surfaces.
- Chapters 12, 14 & 15 of the text, excluding the proof of the theorem 3 and the corollary to theorem 4 in chapter 15

Reference

- Serge Lang, Differential Manifolds
- I.M. Siger, J.A Thorpe, Lecture notes on Elementary topology and Geometry, Springer (1967)
- S. Sternberg, Lectures on Differential Geometry, Prentice-Hall, 1964.
- M. DoCarmo, Differential Geometry of curves and surfaces.
- Goursat, Mathematical Analysis, Vol-1(last two chapters)

CHAPTER 1
LEVEL SETS AND GRAPHS

Level Sets

Definition-Level Set

Given a function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^{n+1}$, its level sets are the sets $f^{-1}(c)$ defined, for each real number c, by

$$f^{-1}(c) = \{(x_1, \ldots, x_{n+1}) \in U \mid f(x_1, \ldots, x_{n+1}) = c\}$$

.

Remarks

- The number c is called the height of the level set, and $f^{-1}(c)$ is called the level set at height c. Since $f^{-1}(c)$ is the solution set of the equation $f(x_1, \ldots, x_{n+1}) = c$

Graph

Definition-Graph

The graph of a function $f : U \to \mathbb{R}$ is the subset of $\mathbb{R}^{n+2}$ defined by

$$\text{graph}(f) = \{(x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} : (x_1, \ldots, x_{n+1}) \in U \& f(x_1, \ldots, x_{n+1}) = x_{n+2}\}$$

Remark

- Level sets can be empty set
- For $c \geq 0$, the level set of f at height c is just the set of all points in the domain of f over which the graph is at distance c.
- For $c < 0$, the level set of f at height c is just the set of all points in the domain of f under which the graph lies at distance $-c$.

Example

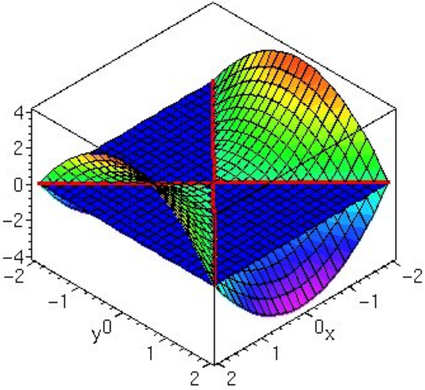
If $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$, then the level set at $c = -1$ is the empty set.

Problems

1. Describe the level sets of $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$, for $n = 0, 1, 2$.
2. Describe the level sets of $f(x_1, x_2) = -x_1^2 + x_2^2$.
3. Describe the level sets of $f(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3$.

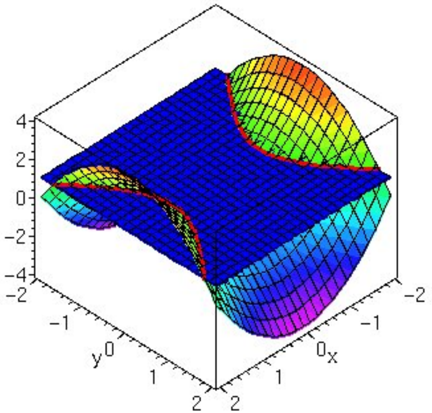
Level curve

Level curve $f(x, y) = x^2 - y^2$ at $c = 0$



Level curve

Level curve of $f(x,y) = x^2 - y^2$ at $c = 1$.



Exercise

Describe the graphs and level sets(level curves) of the following functions

1. $f(x_1, x_2) = x_1$.
2. $f(x_1, x_2) = x_1 - x_2$.
3. $f(x_1, x_2) = x_1^2 - x_2^2$.
4. $f(x_1, x_2) = 3r^8 - 8r^6 + 6r^4$.

CHAPTER 2
VECTOR FIELDS

Vector

Definition-Vector

A vector at a point $p \in \mathbb{R}^{n+1}$ is a pair $v = (p, v)$ where $v \in \mathbb{R}^{n+1}$.

Remarks

- The vectors at p form a vector space $\mathbb{R}^{n+1}_p$ of dimension $n+1$, with addition defined by $(p,v)+(p,w) = (p, v+w)$ and scalar multiplication by$c(p,v) = (p, cv)$.
- The set of all vectors at all points of $\mathbb{R}^{n+1}$ can be identified (as a set) with the Cartesian product $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{2n+2}$.
- Rule of addition does not permit the addition of vectors at different points of $\mathbb{R}^{n+1}$.

Dot product and Cross product

definition-dot product

If $(p,v),(p,w)$ are two vectors at p, their dot product is defined as,

$$(p,v).(p,w) = v.w.$$

definition-cross product

If $(p,v),(p,w) \in \mathbb{R}^3_p$ where $p \in \mathbb{R}^3_p$ are two vectors at p, their cross product is defined as,

$$(p,v) \times (p,w) = v \times w.$$

Vector Fields

Definition-Vector Fields

A vector field $\mathbb{X}$ on $U \subset \mathbb{R}^{n+1}$ is a function which assigns to each point of U a vector at that point.

$$\mathbb{X}(p) = (p, X(p))$$

for some function $X : U \to \mathbb{R}^{n+1}$.

Vector Fields

Definition-Smooth Vector Fields

A function $f : U \to \mathbb{R}$ (U an open set in $\mathbb{R}^{n+1}$) is smooth if all its partial derivatives of all orders exist and are continuous.

A function $f : U \to \mathbb{R}^k$ (U an open set in $\mathbb{R}^{n+1}$) defined by $f(p) = (f_1(p), \ldots, f_k(p))$ is smooth if each component function $f_i : U \to \mathbb{R}$ is smooth.

Definition-Gradient

Associated with each smooth function $f : U \to \mathbb{R}$ (U open in $\mathbb{R}^{n+1}$ is a smooth vector field on U called the gradient ∇f, of f defined by

$$(\nabla f)(p) = \left(p, \frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_{n+1}}(p)\right)$$

Definition- Parametrized Curve

A parametrized curve in $\mathbb{R}^{n+1}$ is a smooth function $\alpha : I \to \mathbb{R}^{n+1}$, where I is some open interval in $\mathbb{R}$. By smoothness of such a function is meant that α is of the form $\alpha(t) = (x_1(t), \ldots, x_{n+1}(t))$ where each x_i is a smooth real valued function on I.

Example-Velocity Vector

The velocity vector at time $t (t \in I)$ of the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ is the vector at $\alpha(t)$ defined by

$$\dot{\alpha}(t) = \left(\alpha(t), \frac{d\alpha}{dt}(t)\right) = \left(\alpha(t), \frac{dx_1}{dt}(t), \ldots, \frac{dx_{n+1}}{dt}(t)\right)$$

Definition-Integral Curve

A parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ is said to be an integral curve of the vector field $\mathbb{X}$ on the open set U in $\mathbb{R}^{n+1}$ if $\alpha(t) \in U$ and $\dot{\alpha}(t) = \mathbb{X}(\alpha(t))$ for all $t \in I$.

Remark

α has the property that its velocity vector at each point of the curve coincides with the value of the vector field at that point

Theorem-Statement

Let $\mathbb{X}$ be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Then there exists an open interval I containing 0 and an integral curve $\alpha : I \to U$ of $\mathbb{X}$ such that

(i) $\alpha(0) = p$

(ii) If $\beta : \tilde{I} \to U$ is any other integral curve of $\mathbb{X}$ with $\beta(0) = p$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in I$

Sketch of the proof

Let $\mathbb{X}(p) = (p, X_1(p), \ldots, X_{n+1}(P))$ be a smooth vector field and let $\alpha(t) = (x_1(t), \ldots, x_{n+1}(t))$ be a parametrized curve where $X_i : U \to \mathbb{R}$, and $x_i : I \to \mathbb{R}$. Then the requirement is $\dot{\alpha}(t) = \mathbb{X}(\alpha(t))$

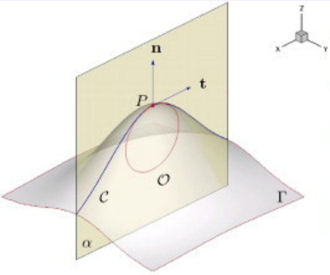
$$\begin{aligned} \frac{dx_1}{dt}(t) &= X_1(x_1(t), \ldots, x_{n+1}(t)) \\ &\vdots \\ \frac{dx_{n+1}}{dt}(t) &= X_{n+1}(x_1(t), \ldots, x_{n+1}(t)) \end{aligned}$$

CHAPTER 3
THE TANGENT SPACE

Tangent

Definition

Let $f : U \to \mathbb{R}$ be a smooth funtion, let $c \in \mathbb{R}$ be such that $f^{-1}(c)$ is non empty, and let $p \in f^{-1}(c)$. A vector at p is said to be tangent to the level set $f^{-1}(c)$ if it is a velocity vector of a parametrized curve in $\mathbb{R}^{n+1}$ whose image is contained in $f^{-1}(c)$.



Lemma

Statement

The gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p.

References

- Chain Rule:

$$\begin{aligned}\frac{d}{dt} f(x_1(t), \ldots, x_{n+1}(t)) &= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_{n+1}}\frac{dx_{n+1}}{dt} \\ &= \nabla f(\alpha(t)) \cdot \dot{\alpha}(t)\end{aligned}$$

$$\begin{aligned}\text{where } \alpha(t) &= (x_1(t), \ldots, x_{n+1}(t)) \text{ and} \\ \dot{\alpha}(t) &= \left(\frac{dx_1}{dt}, \ldots, \frac{dx_{n+1}}{dt}\right)\end{aligned}$$

Proof

For a vector v tangent to $f^{-1}(c)$ at p, there is a parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ with $v = \dot{\alpha}(t_0)$, $\alpha(t_0) = p$, and $Im\, \alpha \subset f^{-1}(c)$

$$Image\, \alpha \subset f^{-1}(c) \implies f(\alpha(t)) = c, \ \forall t \in I.$$

Differentiating, we get

$$0 = \frac{d}{dt}(f \circ \alpha)(t_0) = \nabla f(\alpha(t_0)) \cdot \dot{\alpha}(t_0) = \nabla f(p) \cdot \dot{\alpha}(t_0)$$

Remark

If $\nabla f(p) \neq 0$, the set of all vectors tangent to $f^{-1}(c)$ at p is contained in the n-dimensional vector subspace $[\nabla f(p)]^{\perp}$ of $\mathbb{R}_p^{n+1}$ consisting of all vectors orthogonal to $\nabla f(p)$

Definition-Regular Point

A point $p \in \mathbb{R}^{n+1}$ such that $\nabla f(p) \neq 0$ is called regular point.

Theorem-Characterisation of tangent

Theorem-Statement

Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Then the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

References

- $[\nabla f(p)]^{\perp} = \{\mathbf{v} = (p, v) : \nabla f(p) \cdot \mathbf{v} = 0\}$
- Let $T = \{\mathbf{v} = (p, v) : \mathbf{v}$ is a tangent vector to $f^{-1}(c)$ at $p\}$. Then by previous lemma $T \subseteq [\nabla f(p)]^{\perp}$.
- In order to prove the theorem, it is enough to show that

$$[\nabla f(p)]^{\perp} \subseteq T$$

Proof

Let $\mathbf{v} = (p, v) \in [\nabla f(p)]^{\perp}$. Then $\nabla f(p) \cdot \mathbf{v} = 0$.
Now define a constant vector field $\mathbb{X}(q) = (q, v)\ \forall\, q \in \mathbb{R}^{n+1}$. Clearly $\mathbb{X}(p) = (p, v) = \mathbf{v}$
Using this define a new vector field

$$\mathbb{Y}(q) = \mathbb{X}(q) - \left(\frac{\mathbb{X}(q) \cdot \nabla f(q)}{\|\nabla f(q)\|^2}\right) \nabla f(q)$$

Then Domain of $\mathbb{Y} = U = \{q \in \mathbb{R}^{n+1} : \nabla f(q) \neq 0\}$. Clearly $p \in U$, since p is a regular point.
We have, $\mathbb{Y}(q) \cdot \nabla f(q) = 0\ \forall\, q \in U$ and $\mathbb{Y}(p) = \mathbb{X}(p) = \mathbf{v}$
Now let α be an integral curve of $\mathbb{Y}$ through p.
Then $\alpha(0) = p$ and $\dot{\alpha}(t) = \mathbb{Y}(\alpha(t))$

proof

In particular, $\dot{\alpha}(0) = \mathbb{Y}(\alpha(0)) = \mathbb{Y}(p) = \mathbb{X}(p) = \mathbf{v}$ and

$$\frac{d}{dt} f(\alpha(t)) = \nabla f(\alpha(t)) \cdot \dot{\alpha}(t) = \nabla f(\alpha(t)) \cdot \mathbb{Y}(\alpha(t)) = 0$$

i,e., $f(\alpha(t)) = \text{ a constant } = c$ $[\because f(\alpha(0)) = f(p) = c]$, then

$$\alpha \subset f^{-1}(c)$$

Remark

At each regular point p on a level set $f^{-1}(c)$ of a smooth function there is a well defined tangent space consisting of all velocity vectors at p of all parametrized curves in $f^{-1}(c)$ passing through p, and this tangent space is precisely $[\nabla f(p)]^{\perp}$.

CHAPTER 4
SURFACES

Surfaces

Definition-Surface

A surface of dimension n, or n-surface, in $\mathbb{R}^{n+1}$ is a non-empty subset S of $\mathbb{R}^{n+1}$ of the form $S = f^{-1}(c)$ where $f : U \to \mathbb{R}$, U open in $\mathbb{R}^{n+1}$, is a smooth function with the property that $\nabla f(p) \neq 0$ for all $p \in S$.

Remark

- A 1-surface in $\mathbb{R}^2$ is also called a plane curve.
- A 2-surface in $\mathbb{R}^3$ is usually called simply a *surface*.
- An n-surface in $\mathbb{R}^{n+1}$ is often called a hypersurface, especially when $n > 2$.

Surface and Tangent Space

Each n-surface S has at each point $p \in S$ a tangent space which is an n-dimensional vector subspace of the space $\mathbb{R}^{n+1}_p$ of all vectors at p.

Remark

- The tangent space at the point p on the surface S will be denoted by S_p.
- If f is any smooth function such that $S = f^{-1}(c)$ for some $c \in \mathbb{R}$ and $\nabla f(p) \neq 0$ for all $p \in S$, then S_p may also be described as $[\nabla f(p)]^{\perp}$.

n-Sphere

Example-4.1

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined by $f(x_1, \dots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$. Then $S = f^{-1}(1)$ is an n-surface with $\nabla f(p) \neq 0$, $\forall\, p \in S$ where $\nabla f(p) = (p, 2x_1, \dots, 2x_{n+1})$
This n-surface is called the unit n-sphere

Remark

- For a vector $(p, v) \in \mathbb{R}^{n+1}$ to be zero it is only necessary that $v = 0$
- $$\nabla f(p) = 0 \implies 2x_1 = \cdots = 2x_{n+1} = 0$$ $$\implies (x_1, \dots, x_{n+1}) = (0, \dots, 0) \notin S$$

n-Plane

Example-4.2

For $0 \neq (a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$ and $b \in \mathbb{R}$, the n-plane

$$a_1 x_1 + \cdots + a_{n+1} x_{n+1} = b$$

is the level set $f^{-1}(b)$ where $f(x_1, \ldots, x_{n+1}) = a_1 x_1 + \cdots + a_{n+1} x_{n+1}$. It is an n-surface for each $b \in \mathbb{R}$ since

$$\nabla f(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{n+1}, a_1, \ldots, a_{n+1}) \neq 0$$

Remarks

- A 1-plane is called a line in $\mathbb{R}^2$.
- A 2-plane is called simply a *plane* in $\mathbb{R}^3$.
- An n-plane for $n > 2$ is sometimes called a hyperplane in $\mathbb{R}^{n+1}$.
- Two different values of b with the same value of $(a_1, \ldots, a_{n+1})$ define parallel n-planes.

Example-4.3

Let C be a curve in $\mathbb{R}^2$ which lies above the x_1-axis.
Thus $C = f^{-1}(c)$ for some $f : U \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in C$, where U is contained in the upper half plane $x_2 > 0$. Define $S = g^{-1}(c)$ where $g : U \times \mathbb{R} \to \mathbb{R}$ by $g(x_1, x_2, x_3) = f(x_1, (x_2^2 + x_3^2)^{1/2})$. Then S is a 2-surface.
Each point $p = (a, b) \in C$ generates a circle of points of S, namely the circle in the plane $x_1 = a$ consisting of those points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1 = a$, $x_2^2 + x_3^2 = b$.

Remark

The surface S is called the surface of revolution obtained by rotating the curve C about the x_1-axis

Graph and Surface

Graph as n-Surface

Let $f : U \to \mathbb{R}$ be a smooth function on U, U open in $\mathbb{R}^n$. Then $graph(f) = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = f(x_1, \dots, x_n)\}$ is an n-surface $S = g^{-1}(0)$ in $\mathbb{R}^{n+1}$, where

$$g(x_1, \dots, x_{n+1}) = x_{n+1} - f(x_1, \dots, x_n) \text{ with}$$

$$\nabla g(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n+1}, -\partial f/\partial x_1, \dots, -\partial f/\partial x_n, 1)$$

n-surface from an $(n-1)$-surface

Let S be an $(n-1)$-surface in $\mathbb{R}^n$, given by $S = f^{-1}(c)$, where $f : U \to \mathbb{R}$ (U open in $\mathbb{R}^n$) with $\nabla f(p) \neq 0$ for all $p \in f^{-1}(c)$. Let $g : U_1 \to \mathbb{R}$, where $U_1 = U \times \mathbb{R}$ be defined by $g(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_n)$. Then $g^{-1}(c)$ is an n-surface in $\mathbb{R}^{n+1}$ because

$$\nabla g(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{n+1}, \partial f/\partial x_1, \ldots, \partial f/\partial x_n, 0) \neq 0$$

when $g(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_n) = c$ because $\nabla f(x_1, \ldots, x_n) \neq 0$ whenever $(x_1, \ldots, x_n) \in f^{-1}(c)$. Then the n-surface $g^{-1}(c)$ is called the cylinder over S

Theorem-Statement

Let S be an n-surface in $\mathbb{R}^{n+1}$, $S = f^{-1}(c)$ where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g : U \to \mathbb{R}$ is a smooth function and $p \in S$ is an extreme point of g on S; i.e., either $g(q) \leq g(p)$ for all $q \in S$ or $g(q) \geq g(p)$ for all $q \in S$. Then there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.

Remark

The number λ is called a Lagrange multiplier.

Proof

The tangent space to S at p is $S_p = [\nabla f(p)]^{\perp}$. Hence $S_p^{\perp}$ is the 1-dimensional subspace of $\mathbb{R}^{n+1}$ spanned by $\nabla f(p)$. Then

$$\nabla g(p) = \lambda \nabla f(p) \text{ for some } \lambda \in \mathbb{R} \iff \nabla g(p) \in S_P^{\perp}$$
$$\iff \nabla g(p) \cdot v = 0;\ \forall\, v \in S_p$$

But each $v \in S_p$ is of the form $v = \dot{\alpha}(t_0)$ for some parameterized curve $\alpha : I \to S$ and $t_0 \in I$ with $\alpha(t_0) = p$.

Since p is the extreme point of g on S, t_0 is an extreme point of $g \circ \alpha : I \to \mathbb{R}$. Hence

$$0 = (g \circ \alpha)'(t_0) = \nabla g(p) \cdot \dot{\alpha}(t_0) = \nabla g(p) \cdot v\ \forall\, v \in S_P$$

Remark

If S is compact (closed and bounded), then every smooth function $g : U \to \mathbb{R}$ attains a maximum on S and a minimum on S. The above theorem can then be used to locate candidates for these extreme points.

If S is not compact, there may be no extrema.

CHAPTER 5
VECTOR FIELDS ON SURFACES; ORIENTATION

Definitions

Vector Field on a Surface

A vector field X on an n-surface $S \subseteq \mathbb{R}^{n+1}$ is a function which assigns to each point p in S a vector $X(p) \in \mathbb{R}^{n+1}$ at p.

Tangent vector field

If $X(p)$ is tangent to S (i.e., $X(p) \in S_p$) for each $p \in S$, X is said to be a tangent vector field on S.

Normal vector field

If $X(p)$ is orthogonal to S (i.e., $X(p) \in S_p^{\perp}$) for each $p \in S$, X is said to be a normal vector field on S

Remarks

- We shall work almost exclusively with functions and vector fields which are smooth.
- A function $g : S \to \mathbb{R}^k$, where S is an n-surface in $\mathbb{R}^{n+1}$, is smooth if it is the restriction to S of a smooth function $g : V \to \mathbb{R}^k$ defined on some open set V in $\mathbb{R}^{n+1}$ containing S.
- A vector field $\mathbb{X}$ on S is smooth if it is the restriction to S of a smooth vector field defined on some open set containing S.
- $\mathbb{X}$ is smooth if and only if $X : S \to \mathbb{R}^{n+1}$ is smooth, where $\mathbb{X}(p) = (p, X(p))$ for all $p \in S$.

Theorem-Statement

Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\mathbb{X}$ be a smooth tangent vector field on S, and let $p \in S$. Then there exists an open interval I containing 0 and a parametrized curve $\alpha : I \to S$ such that

(i) $\alpha(0) = p$

(ii) $\dot{\alpha}(t) = \mathbb{X}(\alpha(t))$ for all $t \in I$

(iii) If $\beta : \widetilde{I} \to S$ is any other parametrized curve in S satisfying (i) and (ii), then $\widetilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \widetilde{I}$.

Proof

Since X is smooth, there exists an open set V containing S and a smooth vector field $\mathbb{X}$ on V such that $\widetilde{\mathbb{X}}(q) = \mathbb{X}(q)$ for all $q \in S$. Let $f : U \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $S = f^{-1}(c)$ and $\nabla f(q) \neq 0$ for all $q \in S$. Let $W = \{q \in U \cap V : \nabla f(q) \neq 0\}$.

Then W is an open set containing S, and both $\widetilde{\mathbb{X}}$ and f are defined on W. Let $\mathbb{Y}$ be the vector field on W, everywhere tangent to the level sets of f, defined by

$$\mathbb{Y}(q) = \widetilde{\mathbb{X}}(q) - \left(\frac{\widetilde{\mathbb{X}}(q) \cdot \nabla f(q)}{\|\nabla f(q)\|^2} \right) \nabla f(q)$$

Note that $\mathbb{Y}(q) = \mathbb{X}(q)$ for all $q \in S$, since $\mathbb{X}(q) \cdot \nabla f(q) = 0$, $\forall\, q \in S$. Let $\alpha : I \to W$ be the maximal integral curve of Y through p. Then α actually maps I into S because

$$(f \circ \alpha)'(t) = \nabla f(\alpha(t)) \cdot \dot{\alpha}(t) = \nabla f(\alpha(t)) \cdot \mathbb{Y}(\alpha(t)) = 0$$

This shows that $f \circ \alpha$ is constant with $(f \circ \alpha)(0) = f(p) = c$. So $(f \circ \alpha)(t) = c, \ \forall\, t \in I$. Being α the integral curve of $\mathbb{Y}$ through p, $\alpha(0) = p$ and $\dot{\alpha}(t) = \mathbb{Y}(\alpha(t)) = \mathbb{X}(\alpha(t)), \forall t \in I$. Then (i) & (ii) are satisfied. (iii) follows from the theorem in chapter 2.
This completes the proof.

Corollary-Statement

Let $S = f^{-1}(c)$ be an n-surface in $\mathbb{R}^{n+1}$, where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$, and let $\mathbb{X}$ be a smooth vector field on U whose restriction to S is a tangent vector field on S. If $\alpha : I \to U$ is any integral curve of X such that $\alpha(t_0) \in S$ for some $t_0 \in I$, then $\alpha(t) \in S$ for all $t \in I$.

Corollary-Proof

Suppose $\alpha(t) \notin S$ for some $t \in I,\ t > t_0$. Let t_1 denote the greatest lower bound of the set $\{t \in I : t > t_0 \text{ and } \alpha(t) \notin S\}$

Then $f(\alpha(t)) = c$ for $t_0 \leq t < t_1$ so, by continuity, $f(\alpha(t_1)) = c$; that is, $\alpha(t_1) \in S$.

Let $\beta : \tilde{I} \to S$ be an integral curve through $\alpha(t_1)$ of the restriction of $\mathbb{X}$ to S. Then β is also an integral curve of $\mathbb{X}$, sending 0 to $\alpha(t_1)$, as is the curve $\tilde{\alpha}$ defined by $\tilde{\alpha}(t) = \alpha(t + t_1)$.

By uniqueness of integral curves, $\alpha(t) = \tilde{\alpha}(t - t_1) = \beta(t - t_1) \in S$ for all t such that $t - t_1$ is in the common domain of $\tilde{\alpha}$ and β.

But this contradicts the fact that $\alpha(t) \notin S$ for values of t arbitrarily close to t_1 Hence $\alpha(t) \in S$ for all $t \in I$ with $t > t_0$. The proof for $t < t_0$ is similar.

Definition-Connected Set

A subset S of $\mathbb{R}^{n+1}$ is said to be connected if for each pair p, q of points in S there is a continuous map $\alpha : [a, b] \to S$, from some closed interval $[a, b]$ into S, such that $\alpha(a) = p$ and $\alpha(b) = q$.

Remark

- S is connected if each pair of points in S can be joined by a continuous, but not necessarily smooth, curve which lies completely in S.
- The n-sphere $\left(\left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{k=1}^{n+1} x_k^2 = 1\right\}\right)$ is connected if and only if $n \geq 1$.

Theorem-Statement

Let $S \subseteq \mathbb{R}^{n+1}$ be a connected n-surface in $\mathbb{R}^{n+1}$. Then there exist, on S exactly two smooth unit normal vector fields $\mathbb{N}_1$ and $\mathbb{N}_2$, and $\mathbb{N}_2(p) = -\mathbb{N}_1(p)$ for all $p \in S$.

Proof

Let $f : U \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $S = f^{-1}(c)$ and $\nabla f(p) \neq 0$ for all $p \in S$. Then the vector field $\mathbb{N}_1$ on S defined by

$$\mathbb{N}_1(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}, \quad p \in S$$

clearly has the required properties, as does the vector field $\mathbb{N}_2$ defined by $\mathbb{N}_2(p) = -\mathbb{N}_1(p)$ for all $p \in S$.

Proof

Suppose $\mathbb{N}_3$ were another. Then, for each $p \in S$, $\mathbb{N}_3(p)$ must be a multiple of $\mathbb{N}_1(p)$ since both lie in the $1-$dimensional subspace $S_p^{\perp} \subset \mathbb{R}_p^{n+1}$.

Thus $\mathbb{N}_3(p) = g(p)\mathbb{N}_1(p)$ where $g : S \to \mathbb{R}$ is a smooth function on S ($g(p) = \mathbb{N}_3(p) \cdot \mathbb{N}_1(p)$ for $p \in S$).

Since $\mathbb{N}_1(p)$ and $\mathbb{N}_3(p)$ are both unit vectors, $g(p) = \pm 1$ for each $p \in S$.

Finally, since g is smooth and S is connected, g must be constant on S. Thus either $\mathbb{N}_3 = \mathbb{N}_1$ or $\mathbb{N}_3 = \mathbb{N}_2$

Remarks

- A smooth unit normal vector field on an n-surface S in $\mathbb{R}^{n+1}$ is called an orientation on S.
- Each connected n-surface in $\mathbb{R}^{n+1}$ has exactly two orientations.
- An n-surface together with a choice of orientation is called an oriented n-surface.

- There are subsets of $\mathbb{R}^{n+1}$ which most people would agree should be called n-surfaces but on which there exist no orientations.
- An example is the Mobius band B, the surface in $\mathbb{R}^3$ obtained by taking a rectangular strip of paper, twisting one end through $180°$, and taping the ends together.
- There is no smooth unit normal vector field on B.

- It can be seen by picking a unit normal vector at some point on the central circle and trying to extend it continuously to a unit normal vector field along this circle. After going around the circle once, the normal vector is pointing in the opposite direction!
- Since there is no smooth unit normal vector field on B, B cannot be expressed as a level set $f^{-1}(c)$ of some smooth function $f : U \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$, and hence B is not a 2-surface according to our definition. B is an example of an "unorientable 2-surface".