The Spinning Top

Rigid Bodies

Distance between all pairs of points in the system must remain permanently fixed

Six degrees of freedom:

→ 3 cartesian coordinates specifying position of centre of mass

→ 3 angles specifying orientation of body axes

Orthogonal Transformations

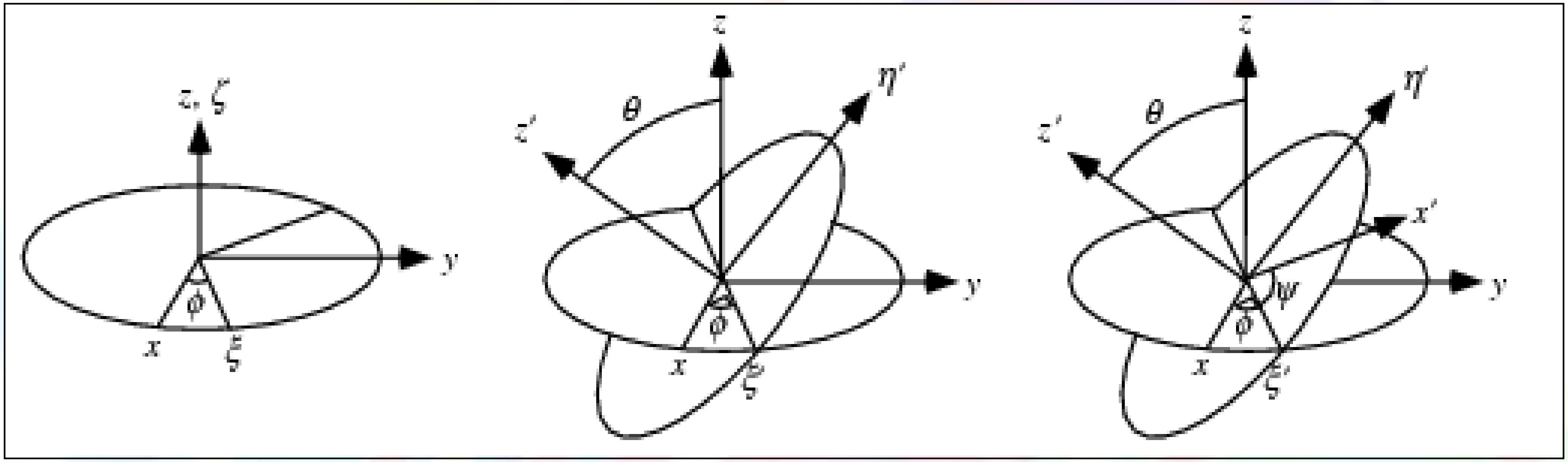
General linear transformation:

matrix of transformation, elements a_{ij}

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

Transition between coordinates fixed in space and coordinates fixed in the rigid body is achieved by means of an orthogonal transformation

Euler Angles



Transformation matrices:

$$\mathbf{D} = \begin{pmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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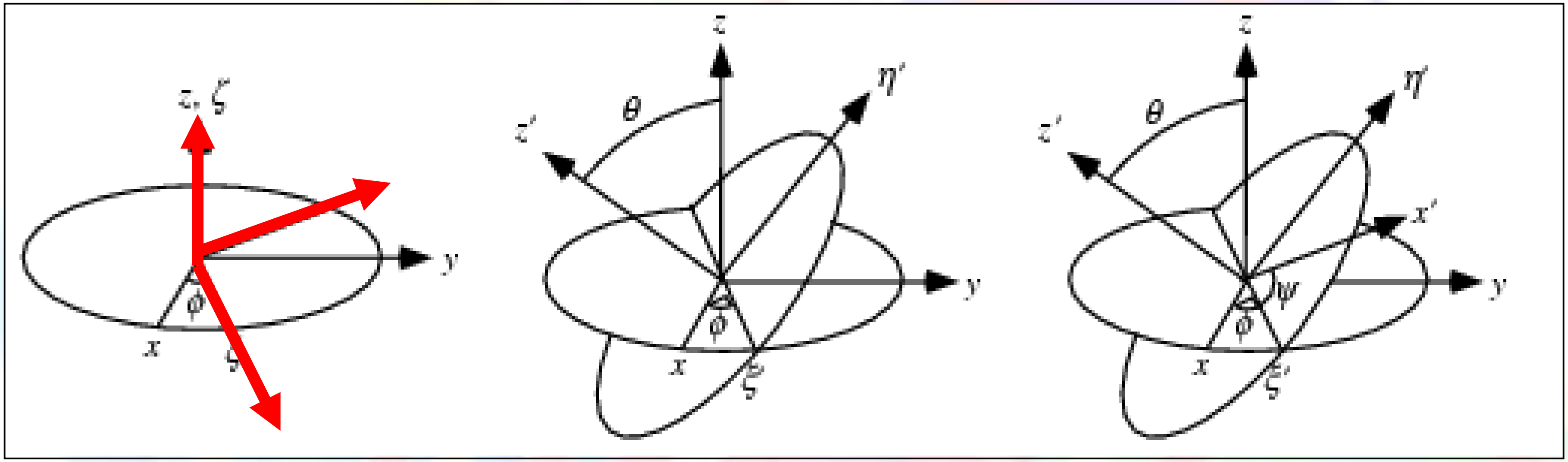
Euler Angles



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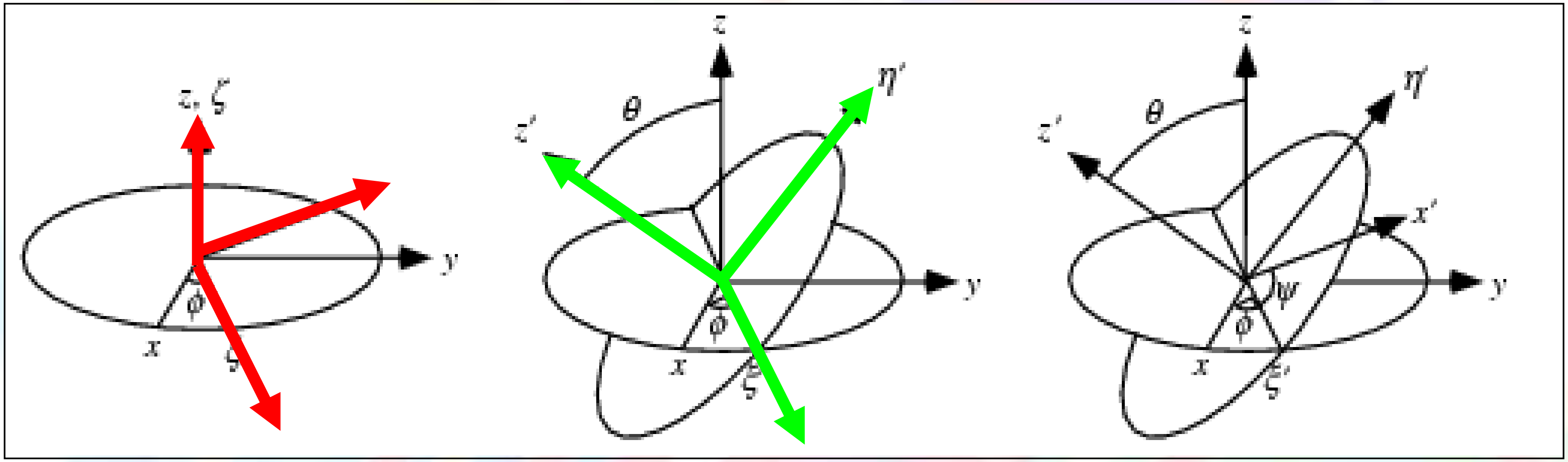

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Euler’s Theorem

“any transformation in the 3-dimensional real space which has at least one fixed point can be described as a simple rotation about a single axis”

Chalses’ Theorem

“the most general displacement of a rigid body is a translation plus a rotation”

Moment of Inertia

Relationship between angular momentum and angular velocity:

$$\mathbf{J} = \underline{\mathbf{I}} \cdot \omega$$

$\underline{\mathbf{I}}$: moment of inertia tensor

$$\underline{\mathbf{I}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Principal moments I_1, I_2, and I_3 found easily if coordinate axes chosen to lie along the directions of the principal axes

Euler's Equations of Motion

For rigid body with one point fixed:

$$I_1\dot{\omega}_x - \omega_y\omega_z(I_2 - I_3) = \tau_x$$

$$I_2\dot{\omega}_y - \omega_z\omega_x(I_3 - I_1) = \tau_y$$

$$I_3\dot{\omega}_z - \omega_x\omega_y(I_1 - I_2) = \tau_z$$

τ : net torque that the body is being subjected to

Force Free Motion of a Rigid Body

Euler's equations for a symmetric body with one point fixed, subject to no net forces or torques:

$$I_1\dot{\omega}_x = (I_1 - I_3)\omega_z\omega_y$$

$$I_2\dot{\omega}_y = -(I_1 - I_3)\omega_z\omega_x$$

$$I_3\dot{\omega}_z = 0$$

Angular frequency:

$$\Omega = \frac{I_1 - I_3}{I_1}\omega_z$$

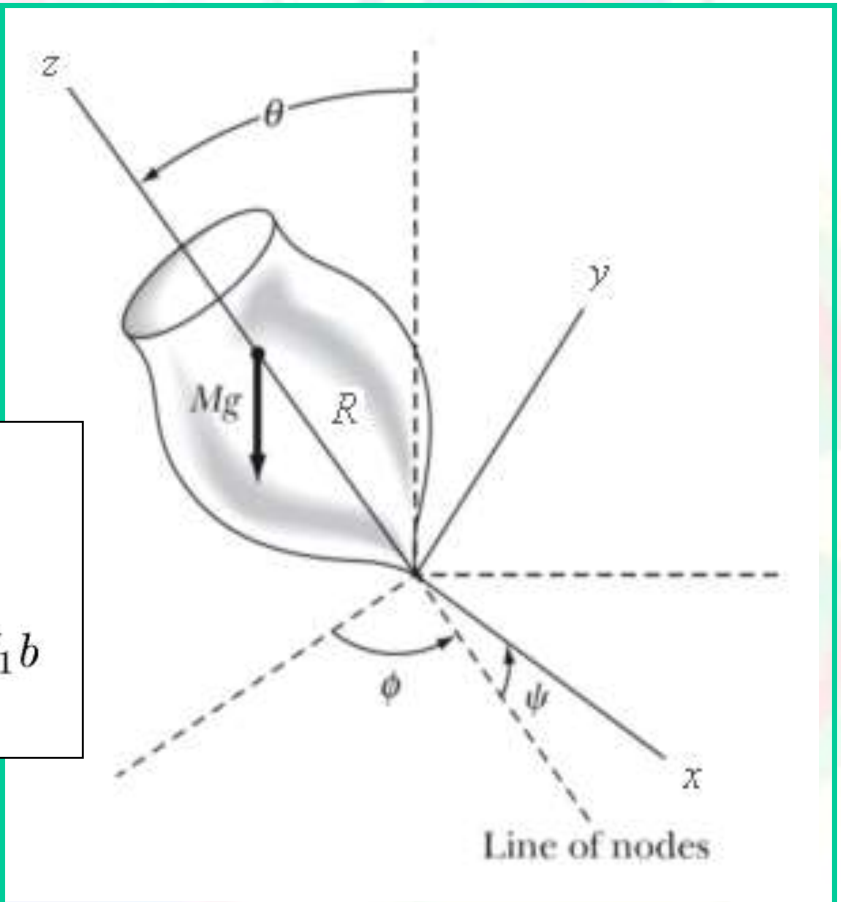
Heavy Symmetrical Top – One Point Fixed

$$L = T - V = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}cos\theta)^2 - MgRcos\theta$$

Generalised momenta corresponding to ignorable coordinates:

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_3\omega_z = I_1 a$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 sin^2\theta + I_3 cos^2\theta)\dot{\phi} + I_3\dot{\psi}cos\theta = I_1 b$$



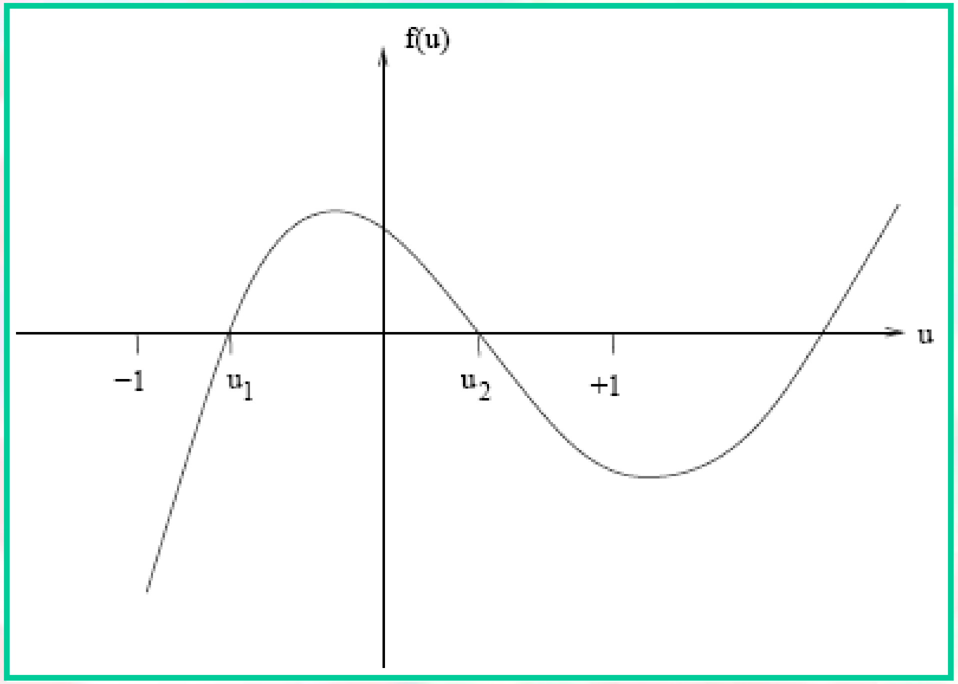
Heavy Symmetrical Top ctd.

Energy equation:

$$\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (a - bu)^2 = f(u)$$

$|f(u)| \rightarrow \infty$ as $u \rightarrow \infty$

$f(\pm 1) = -(b \mp a)^2 \leq 0$



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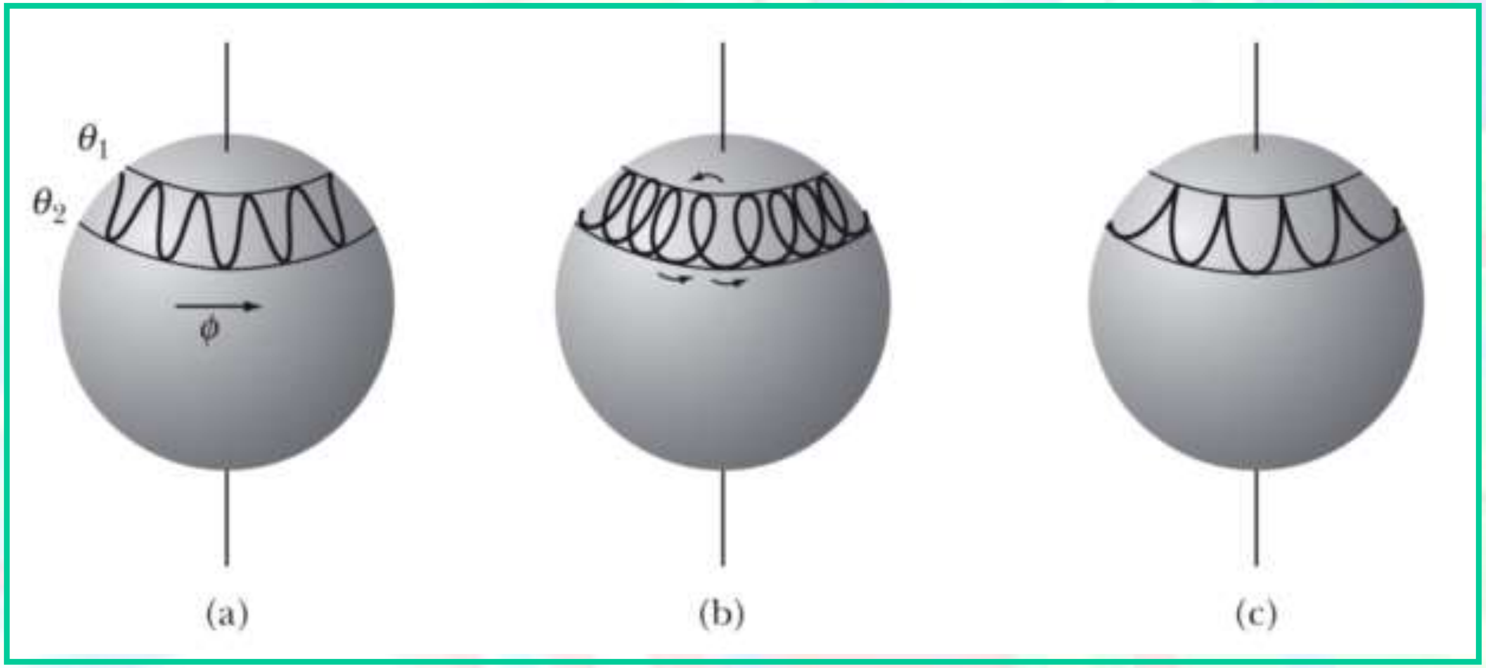
Three possibilities for the motion:



Motion in ϕ: precession

Motion in θ: nutation