Classical Mechanics

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Overview

1. Canonical Transformation

Harmonic Oscillator - an example of canonical transformation

Hamiltonian H for harmonic oscillator -

$$H = \frac{1}{2}kq^2 + \frac{p^2}{2m} \tag{1}$$

k - force constant and $k = m\omega^2$

$$H = \frac{1}{2m}(m^2\omega^2 q^2 + p^2) \tag{2}$$

suggests a transformation in which H is cyclic in the new coordinates. Let us try a canonical transformation of the form

$$p = f(P)\, cos\, Q \tag{3}$$

and

$$q = \frac{f(P)}{m\omega}\, sin\, Q \tag{4}$$

Statement and significance

The Hamiltonian

$$K = H = \frac{1}{2m} f^2(P) \left(cos^2 Q + sin^2 Q \right) = \frac{f^2(P)}{2m} \tag{5}$$

K is cyclic in Q
find f(P)
consider the generating function

$$F_1 = \frac{1}{2} m\omega q^2 cotQ \tag{6}$$

but

$$p = \frac{\partial F_1}{\partial q} = m\omega\, q\, cotQ \tag{7}$$

and

$$P = -\frac{\partial F_1}{\partial Q} = -\frac{1}{2} m\omega\, q^2 cosec^2 Q = \frac{m\omega q^2}{2 sin^2 Q} \tag{8}$$

or

$$q = \sqrt{\frac{2P}{m\omega}}\, sin\, Q \tag{9}$$

substituting q in eq(7)

$$p = \sqrt{2Pm\omega}\, cos\, Q \tag{10}$$

comparing eq(10) with eq(3)

$$f(P) = \sqrt{2Pm\omega} \tag{11}$$

therefore

$$K = \frac{f^2}{2m} = P\omega \tag{12}$$

New Hamiltonian K - cyclic in Q - conjugate momentum P is a constant
H = K = E is the constant energy E
so

$$P = \frac{E}{\omega} \tag{13}$$

$\therefore$ equation of motion transforms to

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \tag{14}$$

The solution of this equation

$$Q = \omega t + \alpha \tag{15}$$

α constant of integration
substituting eq(13) and eq(15) in eq(9)

$$q = \sqrt{\frac{2E}{m\omega^2}}\, sin\,(\omega t + \alpha) \tag{16}$$

and

$$p = \sqrt{2Em}\, cos\,(\omega t + \alpha) \tag{17}$$

Equation (16) and (17) - solutions of Harmonic oscillator