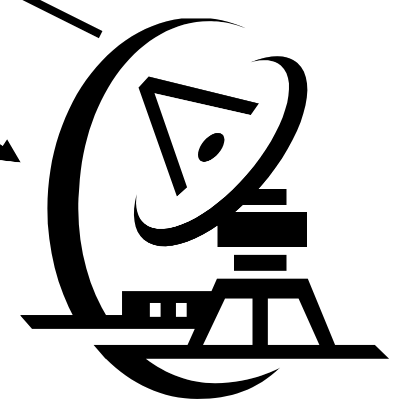
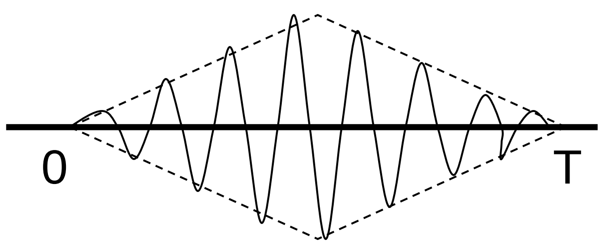
Correlation of Discrete-Time Signals

Transmitted Signal, x(n)

Reflected Signal,

$y(n) = x(n-D) + w(n)$





Cross-Correlation

- Cross-correlation of x(n) and y(n) is a sequence, $r_{xy}(l)$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l) \qquad l = 0, \pm1, \pm2, \ldots$$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n-l)\,y(n) \qquad l = 0, \pm1, \pm2, \ldots$$

- Reversing the order, $r_{yx}(l)$

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\,x(n-l) \qquad l = 0, \pm1, \pm2, \ldots$$

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n-l)\,x(n) \qquad l = 0, \pm1, \pm2, \ldots$$

- => $r_{xy}(l) = r_{yx}(-l)$

Similarity to Convolution

- No folding (time-reversal)

$$r_{xy}\left(l\right) = x\left(l\right) * y\left(-l\right) \qquad r_{yx}\left(l\right) = y\left(l\right) * x\left(-l\right)$$

- In Matlab:
  - Conv(x,fliplr(y))

Auto-Correlation

- Correlation of a signal with itself

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-l) = r_{xx}(-l) \qquad l = 0, \pm 1, \pm 2, \ldots$$

- Used to differentiate the presence of a like-signal, e.g., zero or one
- Even function

Properties

- Two sequences, x(n) and y(n), with finite energy $z(n) = ax(n) + by(n-l)$
- Find energy of z(n)

$$E_z = \sum_{n=-\infty}^{\infty} \left[ax(n) + by(n-l)\right]^2$$

$$= a^2 \sum_{n=-\infty}^{\infty} x^2(n) + b^2 \sum_{n=-\infty}^{\infty} y^2(n-l) + 2ab \sum_{n=-\infty}^{\infty} x(n)\, y(n-l)$$

$$= a^2 r_{xx}(0) + b^2 r_{yy}(0) + 2ab r_{xy}(l) \geq 0$$

E_x E_y

$$E_z = a^2 r_{xx}(0) + b^2 r_{yy}(0) + 2ab r_{xy}(l) \geq 0 \qquad (assume\ b \neq 0)$$

$$= \left(\frac{a}{b}\right)^2 r_{xx}(0) + 2\left(\frac{a}{b}\right) r_{xy}(l) + r_{yy}(0) \geq 0$$

Quadratic in (a/b) and positive, discriminant is non-negative:
For crosscorrelation case:

$$\left|r_{xy}(l)\right| \leq \sqrt{r_{xx}(0)\, r_{yy}(0)} = \sqrt{E_x E_y}$$

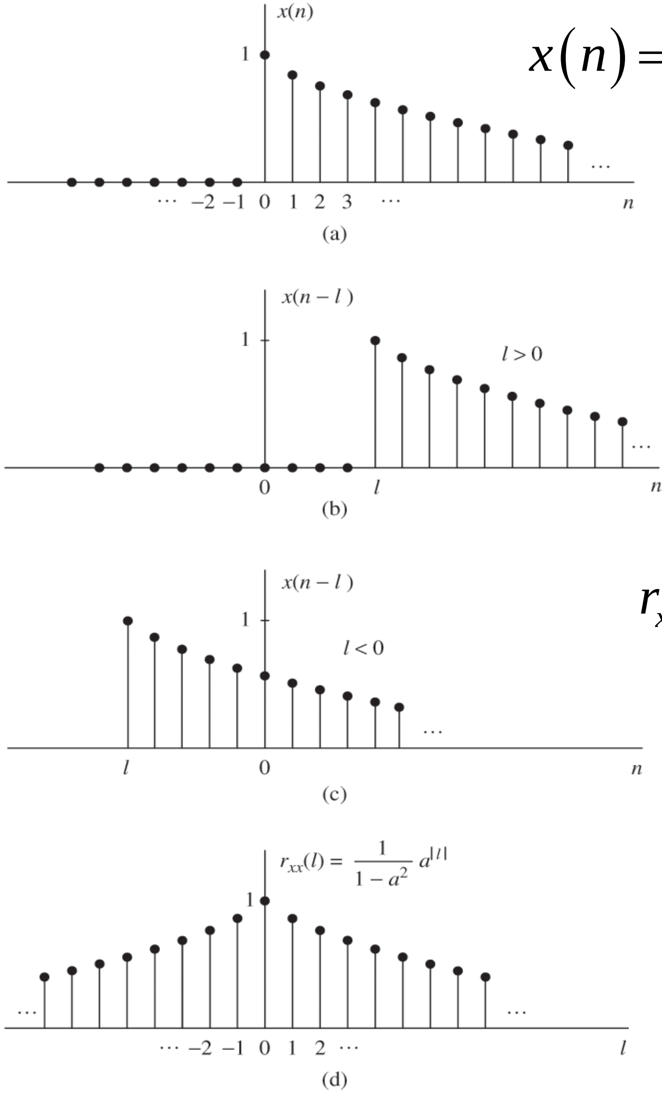
For autocorrelation case:

$$\left|r_{xx}(l)\right| \leq r_{xx}(0) = E_x$$

Maximum value occurs with zero lag (when signals are perfectly matched)
Often normalized to range [-1,1]:

$$\rho_{xx}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}} \qquad \rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)}$$

$$x(n) = a^n u(n); \qquad 0 < a < 1$$



.2 Computation of the autocorrelation of the signal $x(n) = a^n$, $0 < a < 1$.

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-l)$$

$$= \sum_{n=l}^{\infty} a^n a^{n-l} = a^{-l} \sum_{n=l}^{\infty} a^{2n}$$

$$= \frac{1}{1-a^2} a^{|l|} \qquad l \geq 0$$

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)} = a^{|l|} \qquad -\infty < l < \infty$$

Periodic Sequences

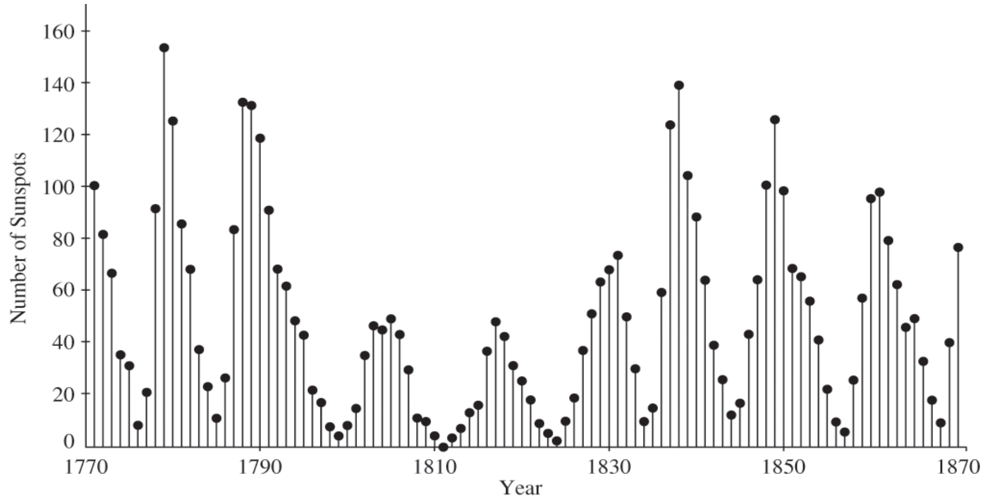
- Power signals crosscorrelation:

$$r_{xy}(l) = \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n) y(n-l)$$

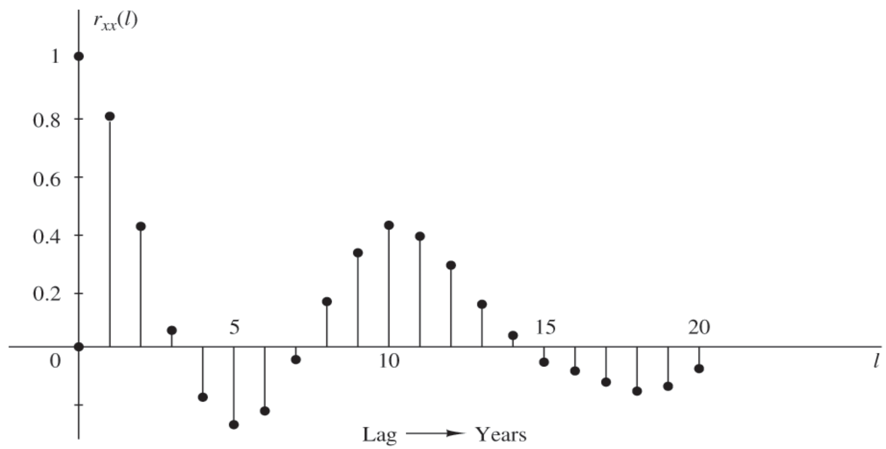
- Define auto and crosscorrelations over one period of the signals
- If x(n) and y(n) are periodic signals with period N:

$$r_{xy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) y(n-l)$$

- Correlations are also periodic with period N



(a)



(b)

Figure 2.6.3 Identification of periodicity in the Wölfer sunspot numbers: (a) annual Wölfer sunspot numbers; (b) normalized autocorrelation sequence.

y(n)=x(n)+w(n)

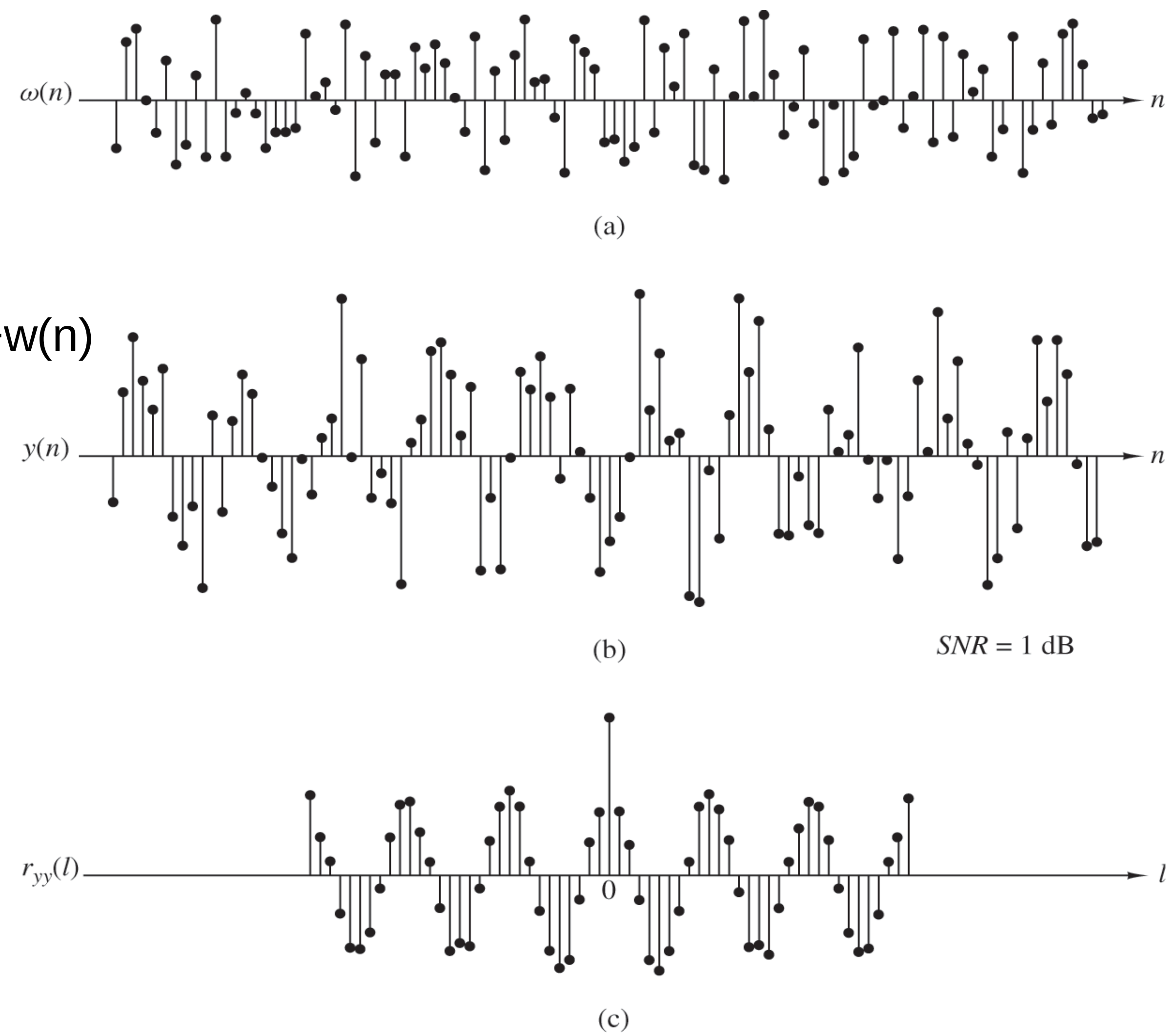


Figure 2.6.4 Use of autocorrelation to detect the presence of a periodic signal corrupted by noise.

LTI Systems

- Convolution, output of LTI system

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

- Crosscorrelation between the output and input signal: $r_{yx}(l) = y(l) * x(-l) = \big(h(l) * x(l)\big) * x(-l)$

$$= h(l) * \big(x(l) * x(-l)\big)$$

$$= h(l) * r_{xx}(l)$$

- Similarly, input to output is:

$$r_{xy}(l) = h(-l) * r_{xx}(l)$$

- Autocorrelation of output:

$$r_{yy}(l) = y(l) * y(-l) = \left(h(l) * x(l)\right) * \left(h(-l) * x(-l)\right)$$
$$= \left(h(l) * h(-l)\right) * \left(x(l) * x(-l)\right)$$
$$= r_{hh}(l) * r_{xx}(l)$$

$$r_{yy}(0) = r_{hh}(0) * r_{xx}(0) = \sum_{k=-\infty}^{\infty} r_{hh}(k) r_{xx}(k)$$